ISOMETRIES, SHIFTS, CUNTZ ALGEBRAS AND MULTIRESOLUTION WAVELET ANALYSIS OF SCALE N

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Abstract. In this paper we show how wavelets originating from multiresolution analysis of scale N give rise to certain representations of the Cuntz algebras $\mathcal{O}_N$, and conversely how the wavelets can be recovered from these representations. The representations are given on the Hilbert space $L^2(\mathbb{T})$ by $(S_i\xi)(z) = m_i(z)\xi(z^N)$. We characterize the Wold decomposition of such operators. If the operators come from wavelets they are shifts, and this can be used to realize the representation on a certain Hardy space over $L^2(\mathbb{T})$. This is used to compare the usual scale-2 theory of wavelets with the scale-N theory. Also some other representations of $\mathcal{O}_N$ of the above form called diagonal representations are characterized and classified up to unitary equivalence by a homological invariant.

Contents

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1. Introduction

Continuing [BJP96], [BrJo96b], [Jor95] we will consider some representations π of the Cuntz algebra $\mathcal{O}_N$ coming from wavelet theory. Our ultimate goal is to establish connections between certain representations of $\mathcal{O}_N$, and their decompositions, and wavelet decompositions for the wavelets arising from multiresolutions with scaling N. The map from wavelets into representations is described in detail in Section 9 (when $N = 2$), Section 10 (when the translates of the father function are orthogonal), and in Section 12 (in more general cases). Unfortunately we have only partial results on how to go the other way, from representations to wavelets, and for the moment the path in both directions leads past certain functions from the circle $\mathbb{T}$ into unitary $N \times N$ matrices given by (1.11). We will discuss further the connection between representations and wavelets at the end of this introduction.

Recall from [Cun77] that $\mathcal{O}_N$ is the C^*-algebra generated by $N \in \mathbb{N}$ isometries $s_0, s_1, \dots, s_{N-1}$ satisfying

$$s_i^* s_j = \delta_{ij} \mathbb{1} \tag{1.1}$$

and

$$\sum_{i=0}^{N-1} s_i s_i^* = \mathbb{1}. \tag{1.2}$$

The representations we will consider are realized on Hilbert spaces $\mathcal{H} = L^2(\Omega, \mu)$ where Ω is a measure space and μ is a probability measure on Ω. We define the representations in terms of certain maps $\sigma_i : \Omega \to \Omega$ with the property that $\mu(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0$ for all $i \neq j$, and if $\rho_i = \mu(\sigma_i(\Omega))$ then $\rho_i > 0$ and $\sum_{i=0}^{N-1} \rho_i = 1$, i.e., $\{\sigma_0(\Omega), \dots, \sigma_{N-1}(\Omega)\}$ is a partition of Ω up to measure zero.

We further assume that

$$\int_\Omega f \, d\mu = \sum_{i=0}^{N-1} \rho_i \int_\Omega f \circ \sigma_i \, d\mu \tag{1.3}$$

for all $f \in L^\infty(\Omega, \mu)$, or, equivalently,

$$\mu(\sigma_i(Y)) = \rho_i \mu(Y) \tag{1.4}$$

for $i \in \mathbb{Z}_N = \{0, 1, \dots, N-1\}$ and all measurable $Y \subset \Omega$. Since $\rho_i > 0$, this entails that the σ_i's are injections up to measure zero, and hence we may define an N-to-1 map $\sigma : \Omega \to \Omega$, well defined up to measure zero, by $\sigma \circ \sigma_i = \mathrm{id}$ for $i \in \mathbb{Z}_N$. Finally, we assume that the sets $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}(\Omega)$ generate the σ-algebra of measurable sets of Ω up to sets of measure zero. Thus (Ω, μ, σ_i) is canonically isomorphic by a coding map to $\Big(\bigtimes_{k=1}^\infty \mathbb{Z}_N$, the product measure of measure on $\mathbb{Z}_N$ with weights $\rho_0, \rho_1, \dots, \rho_{N-1}, \sigma_i^{(0)} \Big)$, where

$$\sigma_i^{(0)}(x_1, x_2, \dots) = (i, x_1, x_2, \dots), \tag{1.5}$$

and then σ is defined by

$$\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots). \tag{1.6}$$

We will list many other realizations of (Ω, μ, σ_i) below.

The announced representations $s_i \to S_i$ of $\mathcal{O}_N$ on $L^2(\Omega, \mu)$ are defined in terms of measurable functions $m_0, m_1, \dots, m_{N-1}$ from Ω into $\mathbb{C}$ with the property that the $N \times N$ matrix

$$\begin{pmatrix} \sqrt{\rho_0} m_0(\sigma_0(x)) & \sqrt{\rho_1} m_0(\sigma_1(x)) & \dots & \sqrt{\rho_{N-1}} m_0(\sigma_{N-1}(x)) \\ \sqrt{\rho_0} m_1(\sigma_0(x)) & \sqrt{\rho_1} m_1(\sigma_1(x)) & \dots & \sqrt{\rho_{N-1}} m_1(\sigma_{N-1}(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\rho_0} m_{N-1}(\sigma_0(x)) & \sqrt{\rho_1} m_{N-1}(\sigma_1(x)) & \dots & \sqrt{\rho_{N-1}} m_{N-1}(\sigma_{N-1}(x)) \end{pmatrix} \tag{1.7}$$

is unitary for almost all $x \in \Omega$. The representation is given by

$$(S_i \xi)(x) = m_i(x) \xi(\sigma(x)) \tag{1.8}$$

and one computes that this is really a representation of the Cuntz algebra and

$$(S_i^* \xi)(x) = \sum_{k \in \mathbb{Z}_N} \rho_k \bar{m}_i(\sigma_k(x)) \xi(\sigma_k(x)); \tag{1.9}$$

see [Jor95]. The computations are (for $\eta, \xi \in L^2(\Omega, \mu)$):

$$\begin{aligned} \langle \eta \,|\, S_i^* \xi \rangle &= \langle S_i \eta \,|\, \xi \rangle \\ &= \int_\Omega \bar{m}_i(x) \bar{\eta}(\sigma(x)) \xi(x) \, d\mu(x) \\ &= \sum_{k=0}^{N-1} \rho_k \int_\Omega \bar{m}_i(\sigma_k(x)) \bar{\eta}(x) \xi(\sigma_k(x)) \, d\mu(x), \end{aligned}$$

which used (1.3) and gives (1.9), and thus

$$\begin{aligned} \left(S_i^* S_j \xi\right)(x) &= \sum_{k \in \mathbb{Z}_N} \rho_k \bar{m}_i(\sigma_k(x)) m_j(\sigma_k(x)) \xi(\sigma(\sigma_k(x))) \\ &= \delta_{ij} \xi(x) \end{aligned}$$

by unitarity of (1.7), which gives (1.1). The formula (1.2) follows similarly from unitarity of (1.7):

$$\sum_i \|S_i^* \xi\|^2 = \sum_i \sum_{k\,l} \rho_k \rho_l \int_\Omega m_i(\sigma_k(x)) \bar{\xi}(\sigma_k(x)) \bar{m}_i(\sigma_l(x)) \xi(\sigma_l(x)) \, d\mu(x).$$

By unitarity of (1.7),

$$\sum_i \sqrt{\rho_k} \bar{m}_i(\sigma_k(x)) \sqrt{\rho_l} m_i(\sigma_l(x)) = \delta_{lk},$$

so

$$\begin{aligned} \sum_i \|S_i^* \xi\|^2 &= \sum_{k\,l} \sqrt{\rho_k} \sqrt{\rho_l} \delta_{lk} \int_\Omega \bar{\xi}(\sigma_k(x)) \xi(\sigma_l(x)) \, d\mu(x) \\ &= \sum_k \rho_k \int_\Omega |\xi(\sigma_k(x))|^2 \, d\mu(x) \\ &= \|\xi\|^2 \end{aligned}$$

and (1.2) follows.

Before surveying the by now rather rich theory of representations of the form (1.8), we will give some alternative descriptions of the system (Ω, μ, σ_i) which are

convenient to use in special circumstances. From now, and through the rest of the paper, we make the simplifying assumption

$$\rho_k = \frac{1}{N} \tag{1.10}$$

for $k \in \mathbb{Z}_N$, although this assumption is easy to remove in many cases. Thus the condition of unitarity is that the $N \times N$ matrix

$$\frac{1}{\sqrt{N}} \begin{pmatrix} m_0(\sigma_0(x)) & m_0(\sigma_1(x)) & \dots & m_0(\sigma_{N-1}(x)) \\ m_1(\sigma_0(x)) & m_1(\sigma_1(x)) & \dots & m_1(\sigma_{N-1}(x)) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1}(\sigma_0(x)) & m_{N-1}(\sigma_1(x)) & \dots & m_{N-1}(\sigma_{N-1}(x)) \end{pmatrix} \tag{1.11}$$

is unitary for almost all $x \in \Omega$, and

$$(S_i \xi)(x) = m_i(x)\, \xi(\sigma(x)), \tag{1.12}$$

$$(S_i^* \xi)(x) = \frac{1}{N} \sum_{k \in \mathbb{Z}_N} \bar{m}_i(\sigma_k(x))\, \xi(\sigma_k(x)). \tag{1.13}$$

Note conversely that if S_i is given by (1.12) (respectively (1.8)) and if the S_i's define a representation of $\mathcal{O}_N$, then the matrix (1.11) (respectively (1.7)) is unitary. Also, as the ranges of the maps $\sigma_0, \dots, \sigma_{N-1}$ are disjoint, any function from $\mathbb{T}$ into unitary matrices has the form (1.11) (respectively (1.7)). Compare this with the well-known fact that if $S_0, \dots, S_{N-1}$ and $T_0, \dots, T_{N-1}$ are any two realizations of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$, there is a unique unitary U on $\mathcal{H}$ such that $S_k = UT_k$, namely $U = \sum_k S_k T_k^*$. Alternatively, if $M_{ij} = T_i^* S_j$, then $S_k = \sum_j T_j M_{jk}$, and $[M_{ij}]$ is a unitary matrix on $\mathcal{H} \otimes \mathbb{C}^N$. Our representations correspond to the special case that

$$(T_i \xi)(x) = \sqrt{N} \chi_{\sigma_i(\Omega)}(x)\, \xi(\sigma(x))$$

and the M_{ij} are multiplication operators defined by $m_i(\sigma_j(\,\cdot\,)) \in L^\infty(\Omega)$.

Here are some equivalent descriptions of (Ω, μ, σ_i). Description 2 will be particularly convenient in connection with the examples coming from wavelets.

Description 1.

$$\begin{aligned} \Omega &= \bigtimes_{k=1}^{\infty} \mathbb{Z}_N, \\ \mu &= \text{Normalized Haar measure}, \\ \sigma_i(x_1, x_2, \dots) &= (i, x_1, x_2, \dots), \\ \sigma(x_1, x_2, \dots) &= (x_2, x_3, \dots). \end{aligned} \tag{1.14}$$

Description 2.

$$\begin{aligned} \Omega &= \mathbb{T} = \text{the unit circle in } \mathbb{C}, \\ \mu &= \text{Normalized Haar measure}, \\ \sigma_k\left(e^{2\pi i\theta}\right) &= \exp\left(2\pi i(\theta + k)/N\right) \text{ when } 0 \le \theta < 1, \\ \sigma(z) &= z^N; \end{aligned} \tag{1.15}$$

so (1.12)–(1.13) take the form

$$(S_i\xi)(z) = m_i(z)\,\xi\left(z^N\right), \tag{1.16}$$

$$(S_i^*\xi)(z) = \frac{1}{N}\sum_{w^N=z} \bar{m}_i(w)\,\xi(w). \tag{1.17}$$

Description 3. This example has an obvious ν-dimensional analogue, replacing N by a $\nu\times\nu$ matrix $\mathbf{N}$ with integer entries such that $|\det(\mathbf{N})| = N$. Then $\Omega = \mathbb{T}^\nu$, $\mu =$ Normalized Haar measure, and $\sigma(x \pmod{\mathbb{Z}^\nu}) = \mathbf{N}x \pmod{\mathbb{Z}^\nu}$ for $x\in\mathbb{R}^\nu$. A somewhat different turn on this idea is the following: let $\mathbf{N}$ be a $\nu\times\nu$ matrix with integer entries such that all the (complex) eigenvalues of $\mathbf{N}$ have modulus greater than 1, and assume $N = |\det(\mathbf{N})|$. Let $D\subset\mathbb{Z}^\nu$ be a set of N points in $\mathbb{Z}^\nu$ which are incongruent modulo $\mathbf{N}(\mathbb{Z}^\nu)$, i.e., such that each point $m\in\mathbb{Z}^\nu$ has a unique expansion $m = d + \mathbf{N}m$ for $d\in D$, $m\in\mathbb{Z}^\nu$. It follows easily [BrJo96b] that there is then a unique compact subset $\mathbf{T}\subset\mathbb{R}^\nu$ such that

$$\mathbf{T} = \mathbf{N}^{-1}\bigcup_{d\in D}(d+\mathbf{T}) \tag{1.18}$$

If μ is Lebesgue measure on $\mathbb{R}^\nu$, we have $\mu(d+\mathbf{T}) = \mu(\mathbf{T})$ for each $d\in D$, while $\mu(\mathbf{N}(\mathbf{T})) = |\det(\mathbf{N})|\,\mu(\mathbf{T}) = N\mu(\mathbf{T})$, and hence the sets $\mathbf{N}^{-1}(d+\mathbf{T})$ must be mutually disjoint up to sets of measure zero, since they have union $\mathbf{T}$. Hence we may define $\Omega = \mathbf{T}$, $\mu =$ Lebesgue measure$|_{\mathbf{T}}$ (except for normalization) and

$$\sigma_i(x) = \mathbf{N}^{-1}(d_i + x) \tag{1.19}$$

where $D = \{d_0, d_1, \dots, d_{N-1}\}$ is an enumeration of D, and

(1.20) $\sigma(x) = y\in\mathbf{T}$ (the point such that there is a $d\in D$ with $x = N^{-1}(d+y)$).

Note $\mathbf{T}$ may or may not be a $\mathbb{Z}^\nu$ tiling of $\mathbb{R}^\nu$, and it may be a union of tiles. An exhaustive discussion of the rich possibilities is given in [BrJo96b], based on [Hut81], [BrJo96a], [JoPe94], [JoPe96].

Description 4. Let $\mathbb{C}$ be the Riemann sphere and let $R(z) = P(z)/Q(z)$ be a rational function, where the polynomials $P(z)$ and $Q(z)$ have no common linear factor. If $N = \max\{\deg P, \deg Q\}$, then R defines an N-fold cover of the Riemann sphere. Now, let Ω be the Julia set, i.e., Ω is the set of $z_0\in\mathbb{C}$ such that the sequence of iterations $R^n(z)$ is not a normal family near z_0, i.e., there is no neighborhood of $z_0\in\mathbb{C}$ such that the sequence $R^n(z)$ is uniformly bounded for z in the neighborhood. It is known that if z_0 is an attracting periodic point, then the boundary of the region of attraction of z_0 under R is equal to Ω [CaGa93, Theorem 2.1], and also that Ω is the closure of the repelling periodic points under R, [CaGa93, Theorem 3.1]. Following [Bro65], [CaGa93], if ν is any probability measure on Ω, define the energy integral

$$I(\nu) = \int_\Omega\int_\Omega \log\left(\frac{1}{|\zeta-\eta|}\right)\,d\nu(\eta)\,d\nu(\zeta). \tag{1.21}$$

Then $\inf_\nu I(\nu) = 0$, and there is a unique probability measure μ such that $I(\mu) = 0$. Furthermore, for a generic set of points $z_0\in\Omega$, the measure μ can be obtained as

the weak limit of the set of probability measures defined by

$$\mu_n = \frac{1}{N^n} \sum_{\substack{w \\ R^n(w)=z_0}} \delta_w. \tag{1.22}$$

Then (Ω, μ, σ) satisfies all our requirements, while σ_i corresponds to an explicit choice of Riemann cover. In the special case $R(z) = z^N$ we recover Description 2.

Let us now describe some known results on the representations defined by (1.12)–(1.13), alias (1.16)–(1.17) (recall that we assume $\rho_i = \frac{1}{N}$ throughout). If

$$m_i(x) = \eta_i^{-1} \chi_{\sigma_i \Omega} \tag{1.23}$$

where $\eta_i \in \mathbb{C}$ are nonzero complex numbers with $\sum_i |\eta_i|^2 = 1$, then $\mathbb{1}$ is cyclic for the representation, and represents the so-called Cuntz state on $\mathcal{O}_N$; see, e.g., [BJP96, Section 8]. In particular these representations are irreducible. In [BJP96, Section 8] we considered the particular representation with

$$m_i(x_0, x_1, x_2, \dots) = N^{\frac{1}{2}} \delta_{ix_0} \langle i, x_1 \rangle, \tag{1.24}$$

where $\langle\ ,\ \rangle$ is the usual Pontrjagin duality $\mathbb{Z}_N \times \hat{\mathbb{Z}}_N \to \mathbb{T}$, and showed that the resulting representation is irreducible and disjoint from all the Cuntz state representations. In [BrJo96a] we considered the representations with

$$m_i(x_0, x_1, \dots) = N^{\frac{1}{2}} \delta_{ix_0} u(x_0, x_1, \dots), \tag{1.25}$$

where $u : \mathbb{T} \to \mathbb{T}$ is a measurable function (using the identification $\Omega = \mathbb{T}$ of Description 2), and we showed in Proposition 7.1 that the resulting representation of $\mathcal{O}_N$ is irreducible, and even the restriction to the canonical UHF-subalgebra $\mathrm{UHF}_N \subset \mathcal{O}_N$ is irreducible. UHF_N is the C^*-subalgebra generated by the monomials $s_I s_J^*$ with $|I| = |J|$. (See also Remark 8.2 of the present paper.) Here $I = (i_1, i_2, \dots, i_n)$ is a finite sequence in $\mathbb{Z}_N$, and $|I| = n$. See [BJP96], [BrJo96a] for details. The significance of the subalgebra UHF_N for our representations derives from the work of Powers on endomorphisms of operator algebras [Pow88]. His endomorphisms correspond to representations of $\mathcal{O}_N$, and the endomorphisms are *shifts* in the sense of Powers iff the corresponding representation is irreducible when restricted to UHF_N. One of the main results of the present paper, Corollary 8.3, states that the irreducible representations obtained from two such functions $u_1, u_2 : \mathbb{T} \to \mathbb{T}$ are unitarily equivalent if and only if there is another measurable function $\Delta : \mathbb{T} \to \mathbb{T}$ such that

$$u_1(z) \Delta(z^N) = \Delta(z) u_2(z). \tag{1.26}$$

In the product space language this relation states that

$$u_1(x_0, x_1, \dots) \Delta(x_1, x_2, \dots) = \Delta(x_0, x_1, \dots) u_2(x_0, x_1, \dots). \tag{1.27}$$

Thus, if for example $u_2 = 1$, we see that some function u_1 of the form

$$u_1(x_0, x_1, \dots) = u_1(x_0, x_1) = \frac{\Delta(x_0)}{\Delta(x_1)} \tag{1.28}$$

will define representations unitarily equivalent to the representation defined by the particular Cuntz state $S_i^* \mathbb{1} = N^{-\frac{1}{2}} \mathbb{1}$. There are of course functions $u_1(x_0, x_1)$ that do not have this form, for example the function $u_1(x_0, x_1, \dots) = \langle x_0, x_1 \rangle$ in [BJP96, Section 8]. This can be used to recover the result from that paper. We use the notation $\langle x_0, x_1 \rangle = \exp\left(i \frac{x_0 x_1 2\pi}{N}\right)$ for $x_0, x_1 \in \mathbb{Z}_N = \{0, 1, \dots, N-1\}$.

In Section 7 we will give an intrinsic characterization of the representations π of $\mathcal{O}_N$ which are given by (1.25). If $\mathcal{D}_N$ is the canonical diagonal C^*-subalgebra of UHF_N, i.e., $\mathcal{D}_N$ is the closure of the linear span of elements of the form $s_I s_I^*$, the characterization up to a decoding of Ω is simply that $\pi\left(\mathcal{D}_N\right)'' \subset M_{L^\infty(\mathbb{T})}$, where $M_{L^\infty(\mathbb{T})}$ is the image of $L^\infty\left(\mathbb{T}\right)$ acting as multiplication operators on $L^2\left(\mathbb{T}\right)$.

In [BrJo96b] and [DaPi96], representations of the form (1.16) with

$$m_i\left(z\right) = \lambda_i z^{d_i} \tag{1.29}$$

were considered, with $\lambda_i \in \mathbb{T}$ and $D = \{d_0, \dots, d_{N-1}\}$ a set of N integers incongruent modulo N. These representations turn out not to be irreducible, but at least when $\lambda_i = 1$ they decompose into a discrete direct sum of mutually disjoint irreducible representations of $\mathcal{O}_N$; and the restriction to UHF_N decomposes similarly [BrJo96b]. When $\lambda_i \neq 1$, even continuous decompositions may occur [DaPi96].

Let us give a more intrinsic characterization of the representations of $\mathcal{O}_N$ given by (1.8).

Proposition 1.1. *Assume that (Ω, μ, σ_i) satisfies the requirements around* (1.3)–(1.4) *and define an endomorphism $\bar\sigma$ of $L^\infty\left(\Omega,\mu\right)$ by $\left(\bar\sigma f\right)\left(x\right) = f\left(\sigma\left(x\right)\right)$. Let $s_i \to S_i$ be a representation of $\mathcal{O}_N$ on $L^2\left(\Omega,\mu\right)$. Then the following conditions are equivalent.*

(1.30) *There are functions $m_i \in L^\infty\left(\Omega\right)$ such that $\left(S_i\xi\right)\left(x\right) = m_i\left(x\right)\xi\left(\sigma\left(x\right)\right)$ for all $\xi \in L^2\left(\Omega,\mu\right)$, $x \in \Omega$.*

(1.31) *$\sum_{i=0}^{N-1} S_i M_f S_i^* = M_{\bar\sigma(f)}$ for all $f \in L^\infty\left(\Omega\right)$, where M_f is the multiplication operator defined by f on $L^2\left(\Omega,\mu\right)$.*

Furthermore, when these conditions are fulfilled, then

$$m_i = S_i \mathbb{1}. \tag{1.32}$$

Proof. (1.30)⇒(1.31). By (1.8) and (1.9) we have, for $f \in L^\infty\left(\Omega\right)$, $\xi \in L^2\left(\Omega,\mu\right)$,

$$\begin{aligned}\sum_{i=0}^{N-1} S_i M_f S_i^* \xi\left(x\right) &= \sum_{i\in\mathbb{Z}_N} m_i\left(x\right)\left(M_f S_i^*\xi\right)\left(\sigma\left(x\right)\right)\\ &= \sum_{i\in\mathbb{Z}_N} m_i\left(x\right) f\left(\sigma\left(x\right)\right)\left(S_i^*\xi\right)\left(\sigma\left(x\right)\right)\\ &= \sum_{i\in\mathbb{Z}_N} m_i\left(x\right) f\left(\sigma\left(x\right)\right) \sum_{k\in\mathbb{Z}_N} \rho_k \bar m_i\left(\sigma_k\sigma\left(x\right)\right)\xi\left(\sigma_k\sigma\left(x\right)\right).\end{aligned}$$

Now, let $k_x \in \mathbb{Z}_N$ be the unique (for almost all x) number such that $x = \sigma_{k_x}\sigma\left(x\right)$. By unitarity of (1.7) for $x := \sigma\left(x\right)$ we have

$$\sum_{i\in\mathbb{Z}_N} m_i\left(x\right)\bar m_i\left(\sigma_k\sigma\left(x\right)\right) = \begin{cases}\rho_{k_x}^{-1} & \text{if } k = k_x\\ 0 & \text{otherwise}\end{cases}$$

and hence (1.31) follows.

(1.31)⇒(1.30). Put

$$m_i = S_i\mathbb{1}.$$

If $f \in L^{\infty}(\Omega)$, we have

$$\begin{aligned} M_{\tilde{\sigma}(f)} S_j &= \sum_{i \in \mathbb{Z}_N} S_i M_f S_i^* S_j \\ &= S_j M_f \end{aligned}$$

and applying this to $\mathbb{1}$ we have

$$f(\sigma(x))\, m_j(x) = (S_j f)(x) .$$

As $L^{\infty}(\Omega)$ is dense in $L^2(\Omega)$, this implies (1.30). $\square$

Let us remark that not all representations of $\mathcal{O}_N$ on a separable Hilbert space $\mathcal{H}$ have the form (1.12) for a suitable realization of $\mathcal{H}$ as $L^2(\Omega, \mu)$. We will for example establish in Theorem 3.1 that the unitary parts of the Wold decompositions of the respective generators S_i have to be zero- or one-dimensional: and, in the case that the representation comes from a wavelet, they have to be zero-dimensional by Lemma 9.3. This is already a severe restriction, which for example immediately implies that none of the representations coming from monomials m_i on $\mathbb{T}$ considered in [BrJo96b] comes from a wavelet! Since we cannot really characterize abstractly the representations of $\mathcal{O}_N$ coming from wavelets, we can of course also not find a completely general way of going the other way, from representations to wavelets. But let us mention some connections from representations to wavelets which are as direct as possible with our present technology: if φ is a father wavelet in $L^2(\mathbb{R})$ satisfying the standard requirements (10.1)–(10.3) in scale N, and $\psi_1, \dots, \psi_{N-1}$ are corresponding mother wavelets as in Theorem 10.1, then any $\xi \in L^2(\mathbb{R})$ has an orthonormal decomposition

$$\xi(\,\cdot\,) = \sum_{i=1}^{N-1} \sum_{j,k \in \mathbb{Z}} a_{jk}^{(i)}(\xi)\, N^{-\frac{j}{2}} \psi_i\left(N^{-j}\,\cdot\, - k\right) \tag{1.33}$$

in $L^2(\mathbb{R})$. In particular ξ is contained in the closed subspace $\mathcal{V}_0$ of $L^2(\mathbb{R})$ spanned by the translates $\varphi(\,\cdot\, - k)$, $k \in \mathbb{Z}$, of the father wavelets if and only if $a_{jk}^{(i)}(\xi) = 0$ for all $j \leq 0$, and in that case φ has also a representation

$$\hat{\xi}(t) = f(t)\, \hat{\varphi}(t) \tag{1.34}$$

in terms of an $f \in L^2(\mathbb{T}) = L^2(\mathbb{R}/2\pi\mathbb{Z})$, where $\hat{}$ denotes Fourier transform (9.6). See Lemma 12.1 for this. The link between the representation and the wavelet formulation is then provided by

$$a_{jk}^{(i)}(\xi) = \left(S_i^* S_0^{*\,j-1} f\right)^{\sim}(k) \tag{1.35}$$

where $(\;)^{\sim}$ refers to the Fourier transform on $L^2(\mathbb{T})$:

$$\tilde{g}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} g(t)\, dt. \tag{1.36}$$

The formula (1.35) follows from Corollary 10.3 and Theorem 6.2, and the details of the proof will be given in Corollary 10.4. So at least given the father wavelet φ, the formula (1.34)–(1.35) give a path from the representation of $\mathcal{O}_N$ to the wavelet $\psi_1, \dots, \psi_{N-1}$. Furthermore, recall that the father wavelet φ under mild regularity

assumptions can be recovered from the function m_0 via the Mallat algorithm (11.3) (see [Mal89], [Dau92]), i.e.,

$$\hat{\varphi}(t) = (2\pi)^{-\frac{1}{2}} \prod_{k=1}^{\infty} \left(N^{-\frac{1}{2}} m_0 \left(tN^{-k} \right) \right). \tag{1.37}$$

But iterating (1.16) n times, and applying the scaling operator

$$(U_N \xi)(t) = N^{-\frac{1}{2}} \xi \left(N^{-1} t \right),$$

we see that

$$(U_N^n S_0^n \xi)(t) = \prod_{k=1}^{n} \left(N^{-\frac{1}{2}} m_0 \left(tN^{-k} \right) \right) \xi(t) \tag{1.38}$$

when functions in $L^2(\mathbb{T})$ are viewed as 2π-periodic functions on $\mathbb{R}$, and taking the limit $n \to \infty$ and using (1.37) we see

$$\lim_{n\to\infty} (U_N^n S_0^n \xi)(t) = (2\pi)^{\frac{1}{2}} \hat{\varphi}(t) \xi(t). \tag{1.39}$$

Thus, when the representation $\{S_i\}$ of $\mathcal{O}_N$ on $L^2(\mathbb{T})$ is given, the formulae (1.39), (1.34), and (1.35) in succession give a prescription for recovering the multiresolution wavelet theory from the representation. Similarly

$$\lim_{n\to\infty} \left(U_N^n S_0^{n-1} S_k \xi \right)(t) = (2\pi)^{\frac{1}{2}} \hat{\psi}_k(t) \xi(t). \tag{1.40}$$

The formulae (1.35), (1.39), and (1.40) were derived under the assumption that the representation of $\mathcal{O}_N$ comes from a wavelet. More fundamentally, if a representation of $\mathcal{O}_N$ is given, Proposition 1.1 gives a necessary and sufficient condition that it defines functions $m_i : \mathbb{T} \to \mathbb{C}$ with the unitarity property (1.11). If we further assume that $m_0(0) = \sqrt{N}$ and m_0 is Lipschitz continuous at 0, then the product expansion (1.37) converges and defines the function $\hat{\varphi}$. But this is still not sufficient for φ to be the father function of a wavelet, as shown by the example between (6.2.4) and (6.2.5) in [Dau92]. If

$$m_0(t) = \sum_{k\in\mathbb{Z}} a_k e^{-ikt} \tag{1.41}$$

is the Fourier expansion of m_0 with $z = e^{-it}$, put

$$m_0^{(k)}(z) = m_0(z) m_0\left(z^N\right) \cdots m_0\left(z^{N^{k-1}}\right). \tag{1.42}$$

Assume now also that m_0 is infinitely differentiable. It is then easy to show that φ is a father function for a wavelet, i.e., (10.1)–(10.3) are valid, if and only if (10.1) alone is valid, i.e.,

$$\{\varphi(\,\cdot\, - k)\}_{k\in\mathbb{Z}} \quad \text{is an orthonormal set.} \tag{1.43}$$

Furthermore, this is again equivalent to any of the following properties (1.44)–(1.47).

(1.44) $\|\varphi\|_{L^2(\mathbb{R})} = 1$.

(1.45) The probability measures $\left| m_0^{(k)}(z) \right|^2 \frac{|dz|}{2\pi}$ converge weakly to Dirac's delta measure on $1 \in \mathbb{T}$.

(1.46) There is a compact set K of reals, congruent to $[-\pi, \pi]$ modulo 2π, such that K contains 0 in its interior and $\hat{\varphi}(t) \neq 0$ for $t \in K$.

The last condition (1.46) is due to A. Cohen [Coh90], and the equivalence of the other two conditions is due to Meyer and Paiva [MePa93]. The latter paper contains an excellent discussion and also shows that these conditions are equivalent to φ being the unique fixed point of the map

$$\psi \rightarrow N^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} a_k \psi\left(N\,\cdot\,+k\right) \tag{1.47}$$

among a regular class of functions satisfying $\hat{\psi}\left(2\pi k\right) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\hat{\psi}\left(0\right) = \left(2\pi\right)^{-\frac{1}{2}}$, this fixed point being an attractor.

Note finally that the condition (1.45) can be translated into the following necessary and sufficient condition that an S_0 of the form (1.16) comes from the father function of a wavelet:

$$\lim_{n\rightarrow\infty} \left\langle S_0^n \mathbb{1} \,\middle|\, M_f S_0^n \mathbb{1} \right\rangle = f\left(0\right) \tag{1.48}$$

for all $f \in C\left(\mathbb{T}\right) = C\left(\mathbb{R} \diagup 2\pi\mathbb{Z}\right)$, where M_f denotes the operator of multiplication by f on $L^2\left(\mathbb{T}\right)$. See Section 12 for details.

2. Cohomology of the map $z \mapsto z^N$ with values in a topological group G

The terminology we introduce in this section will be used in two connections. In Section 3, with $G = \mathbb{T}$, it will be used in the characterization of the unitary part of the Wold decomposition of S_0. In Section 8, also with $G = \mathbb{T}$, it will be used in the characterization of unitary equivalence of two diagonal representations. In Sections 4 and 5 somewhat similar terminology, but with a more general notion of cohomology which is less direct to formulate in the abstract framework, will be used in the case $G = U\left(N\right)$; see for example (4.13). G is a topological group throughout this section.

Let Ω be a measure space, $\sigma : \Omega \rightarrow \Omega$ an endomorphism, and μ a probability measure on Ω such that $\mu\left(\sigma^{-1}\left(Y\right)\right) = \mu\left(Y\right)$ for all measurable $Y \subset \Omega$. Extending the terminology in [CFS82] from the case of automorphisms to the case of endomorphisms, we may define a *cocycle* for σ with values in G as a map $c : \Omega \times \mathbb{N} \rightarrow G$ such that

$$c\left(x, m+n\right) = c\left(x, m\right) c\left(\sigma^m\left(x\right), n\right). \tag{2.1}$$

But then it follows by induction that

$$c\left(x, m\right) = \begin{cases} 1 & \text{if } m = 0 \\ \prod_{k=0}^{m-1} \sigma\left(\sigma^k\left(x\right), 1\right) & \text{if } m > 0 \end{cases} \tag{2.2}$$

so we may and will simply consider a cocycle to be a measurable map $c\left(\,\cdot\,\right) = c\left(\,\cdot\,, 1\right)$ from Ω into G. Any such map defines a proper cocycle through the formula above.

We say that two cocycles c_1, c_2 are *cohomological* if there is a function $\Delta : \Omega \rightarrow G$ such that

$$c_1\left(x\right) = \Delta\left(\sigma\left(x\right)\right)^{-1} c_2\left(x\right) \Delta\left(x\right) \tag{2.3}$$

or

$$c_1\left(x, m\right) = \Delta\left(\sigma^m\left(x\right)\right)^{-1} c_2\left(x, m\right) \Delta\left(x\right). \tag{2.4}$$

In the case that G is abelian, this is the same as saying that c_1 and c_2 *cobound,* i.e., that $c_1\left(x\right) c_2\left(x\right)^{-1}$ is a coboundary. We also say that c_1 and c_2 are cohomologous.

We say in general that a cocycle c is a *coboundary* if there is another cocycle Δ such that

$$c(x) = \Delta(x)\,\Delta(\sigma(x))^{-1} \tag{2.5}$$

or

$$c(x,m) = \Delta(x)\,\Delta(\sigma^m(x))^{-1}. \tag{2.6}$$

In the case that G is abelian, we see that the relation of cohomology is an equivalence relation.

The question of which cocycles are coboundaries is in general a difficult one. Recall for example from [Jor95, Theorem 6.1] that if $c : \mathbb{T} \to \mathbb{T}$ is a Hardy function (i.e., $c \in H^\infty(\mathbb{T})$), and $\sigma(z) = z^2$ for $z \in \mathbb{T}$, then the equation

$$c(z)\,\Delta\left(z^2\right) = \Delta(z) \tag{2.7}$$

has a nonzero solution $\Delta \in L^2(\mathbb{T})$ if and only if c is a monomial, $c(z) = z^n$, and then $\Delta(z) = dz^{-n}$ for a constant d. Another criterion which is more indirect is in [Wal96, Corollary 3]. The version of this corollary which is interesting for us is the following: if c is a measurable cocycle for $z \mapsto z^N$ with values in $\mathbb{T}$, then the following conditions (2.8) and (2.9) are equivalent.

(2.8) There is an $f \in L^\infty(\mathbb{T})$ such that the sequence has a nonzero w*-limit point as $m \to \infty$.

(2.9) The cocycle c is a coboundary.

Furthermore, if these conditions are fulfilled, the cocycle Δ having c as coboundary is unique up to a phase factor, and

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} f\left(z^{N^k}\right) \bar{c}\left(z^{N^{k-1}}\right) \cdots \bar{c}(z) = \bar{\Delta}(z) \int_{\mathbb{T}} f(\eta)\,\Delta(\eta)\,\frac{|d\eta|}{2\pi}$$

in $L^2(\mathbb{T})$, and also pointwise for almost all z, for all $f \in L^\infty(\mathbb{T})$.

The main input in the proof is of course Birkhoff's ergodic theorem, which immediately gives the implication from (2.9) to the conclusion.

3. The Wold decomposition of isometries S_m of the form $(S_m\xi)(z) = m(z)\,\xi\left(z^N\right)$

Equip the circle $\mathbb{T}$ with Haar measure $\frac{|dz|}{2\pi}$, and let $N \in \{2, 3, \dots\}$. Formula (1.3) now takes the form

$$\begin{aligned} \int_{\mathbb{T}} f(z)\,\frac{|dz|}{2\pi} &= \int_{\mathbb{T}} \frac{1}{N} \sum_{\substack{w \\ w^N = z}} f(w)\,\frac{|dz|}{2\pi} \\ &= \int_{\mathbb{T}} f\left(z^N\right) \frac{|dz|}{2\pi}. \end{aligned} \tag{3.1}$$

Let $m : \mathbb{T} \to \mathbb{C}$ be a measurable function. Define $S_m : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$(S_m\xi)(z) = m(z)\,\xi\left(z^N\right). \tag{3.2}$$

We have already computed in (1.17) that

$$(S_m^*\xi)(z) = \frac{1}{N} \sum_{\substack{w \\ w^N = z}} \bar{m}(w)\,\xi(w) \tag{3.3}$$

and hence

$$(S_m^* S_m \xi)(z) = \frac{1}{N} \left(\sum_{\substack{w \\ w^N = z}} |m(w)|^2 \right) \xi(z) . \tag{3.4}$$

It follows immediately from this spectral representation of $S_m^* S_m$ that S_m is bounded if and only if $m \in L^\infty(\mathbb{T})$ and then

$$\|S_m\|^2 = \frac{1}{N} \operatorname*{ess\,sup}_{z \in \mathbb{T}} \left(\sum_{\substack{w \\ w^N = z}} |m(w)|^2 \right) \leq \|m\|_\infty^2 \tag{3.5}$$

Furthermore, we see directly from the spectral representation (3.4) that S_m is an isometry if and only if

$$\frac{1}{N} \sum_{\substack{w \\ w^N = z}} |m(w)|^2 = 1 \tag{3.6}$$

for almost all z.

In general, if S is an isometry, define a decreasing sequence of projections by

$$E_k = S^k S^{*\,k} \tag{3.7}$$

and let

$$P_U = \operatorname*{s\text{-}lim}_{k \to \infty} E_k. \tag{3.8}$$

Then $SP_U = P_U S$, $P_U S$ is a unitary operator on $P_U \mathcal{H}$, and $(1 - P_U) S$ is a shift on $(1 - P_U) \mathcal{H}$, i.e.,

$$\bigcap_n S^n (1 - P_U) \mathcal{H} = \{0\} . \tag{3.9}$$

(Note that the two-sided shift is not a shift with this terminology.) The decomposition

$$S = SP_U \oplus S(1 - P_U) \tag{3.10}$$

is the so-called Wold decomposition of S into a unitary operator and a shift. (For more details on the general Wold decomposition, and some of its applications, the reader is referred to [SzFo70], which also serves as an excellent background reference for the operator theory used in the present paper.) For S_m given by (3.2), a calculation now shows that

$$(E_k \xi)(z) = m^{(k)}(z) \frac{1}{N^k} \sum_{\substack{w \\ w^{N^k} = z^{N^k}}} \bar{m}^{(k)}(w) \xi(w) \tag{3.11}$$

where

$$m^{(k)}(z) = \prod_{j=0}^{k-1} m\left(z^{N^j}\right) . \tag{3.12}$$

Our main result on the Wold decomposition of S_m is the following:

Theorem 3.1. *The projection P_U corresponding to the unitary part of the Wold decomposition of the isometry S_m is one- or zero-dimensional. Furthermore, P_U is one-dimensional if and only if both conditions* (3.13) *and* (3.14) *are fulfilled.*

(3.13) $|m(z)| = 1$ *for almost all* $z \in \mathbb{T}$.

(3.14) *There exists a measurable function* $\xi : \mathbb{T} \to \mathbb{T}$ *and a* $\lambda \in \mathbb{T}$ *such that* $m(z)\,\xi\left(z^N\right) = \lambda \xi(z)$ *for almost all* $z \in \mathbb{T}$.

In this case the range of the projection P_U is $\mathbb{C}\xi \subset L^2(\mathbb{T})$.

In short, S_m is a shift if and only if there exists no phase factor λ such that $\bar{\lambda} m_0$ is a coboundary for the $z \mapsto z^N$ action with values in $\mathbb{T}$.

In order to prove Theorem 3.1, it will be useful to work with the root mean operator $R = R_m$ defined on measurable functions $\xi : \mathbb{T} \to \mathbb{C}$ as follows:

$$(R\xi)(z) = \frac{1}{N} \sum_{\substack{w \\ w^N = z}} |m(w)|^2 \, \xi(w). \tag{3.15}$$

It follows immediately from (3.6) that R is bounded as an operator from $L^p(\mathbb{T})$ into $L^p(\mathbb{T})$ for $1 \le p \le \infty$, and also R preserves positive functions and, from (3.6),

$$R\mathbb{1} = \mathbb{1}. \tag{3.16}$$

Thus

$$\|R\|_{\infty \to \infty} = 1 \tag{3.17}$$

and a computation like the one after (1.8)–(1.9) shows

$$(R^*\xi)(z) = |m(z)|^2 \, \xi\left(z^N\right). \tag{3.18}$$

If $f \in L^\infty(\mathbb{T})$, again let $M_f : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denote the operation of multiplication by f,

$$(M_f \xi)(z) = f(z)\,\xi(z) \tag{3.19}$$

for $\xi \in L^2(\mathbb{T})$.

We will need the formula

$$\begin{aligned} E_k M_f E_k &= M_{(R^k f)\left(z^{N^k}\right)} E_k \\ &= M_{(R^k f)\circ \sigma^k} E_k \end{aligned} \tag{3.20}$$

which follows from (3.11) and (3.15) by the following computation:

$$\begin{aligned} (E_k M_f E_k \xi)(z) &= m^{(k)}(z) \frac{1}{N^k} \sum_{\substack{w \\ w^{N^k} = z^{N^k}}} \bar{m}^{(k)}(w)\, f(w)\, (E_k \xi)(w) \\ &= m^{(k)}(z) \frac{1}{\left(N^k\right)^2} \sum_{\substack{w,v \\ w^{N^k} = z^{N^k} \\ = v^{N^k}}} \bar{m}^{(k)}(w)\, f(w)\, m^{(k)}(w)\, \bar{m}^{(k)}(v)\, \xi(v) \\ &= \left(\frac{1}{N^k} \sum_{\substack{w \\ w^{N^k} = z^{N^k}}} \left| m^{(k)}(w) \right|^2 f(w) \right) \left(\frac{1}{N^k} m^{(k)}(z) \sum_{\substack{v \\ v^{N^k} = z^{N^k}}} \bar{m}^{(k)}(v)\, \xi(v) \right) \\ &= \left(R^k f\right) \left(z^{N^k}\right) (E_k \xi)(z). \end{aligned}$$

Lemma 3.2. *Assume that $P_U \neq 0$ and pick $\xi \in P_U L^2(\mathbb{T})$ such that $\|\xi\|_2 = 1$. It follows that*

$$|\xi(z)|^2 = \lim_{k\to\infty} \prod_{n=0}^{k} \left| m\left(z^{N^n}\right)\right|^2 \tag{3.21}$$

and

$$|\xi(z)| = 1 = |m(z)| \tag{3.22}$$

for almost all $z \in \mathbb{T}$.

Proof. As $P_U\xi = \xi$ and $P_U \leq E_k$, we have $E_k\xi = \xi$ for all $k \in \mathbb{N}$, and using (3.20) on an arbitrary $f \in L^\infty(\mathbb{T})$ we have

$$\begin{aligned}
\int_{\mathbb{T}} f(z)\,|\xi(x)|^2 \frac{|dz|}{2\pi} &= \langle \xi \mid M_f \xi\rangle \\
&= \langle E_k\xi \mid M_f E_k \xi\rangle \\
&= \langle \xi \mid E_k M_f E_k \xi\rangle \\
&= \langle \xi \mid M_{(R^k f)\left(z^{N^k}\right)} \xi\rangle \\
&= \int_{\mathbb{T}} |\xi(z)|^2 \left(R^k f\right)\left(z^{N^k}\right) \frac{|dz|}{2\pi} \\
&= \int_{\mathbb{T}} \frac{1}{N^k} \sum_{\substack{w \\ w^{N^k}=z}} |\xi(w)|^2 \left(R^k f\right)(z) \frac{|dz|}{2\pi}.
\end{aligned}$$

Now use

$$\left(R^k f\right)(z) = \frac{1}{N^k} \sum_{\substack{v \\ v^{N^k}=z}} \left| m^{(k)}(v)\right|^2 f(v) \tag{3.23}$$

to compute further

$$\begin{aligned}
\int_{\mathbb{T}} f(z)\,|\xi(z)|^2 \frac{|dz|}{2\pi} &= \int_{\mathbb{T}} \frac{1}{N^{2k}} \sum_{\substack{w,v \\ w^{N^k}=z \\ v^{N^k}=z}} |\xi(w)|^2 \left| m^{(k)}(v)\right|^2 f(v) \frac{|dz|}{2\pi} \\
&= \int_{\mathbb{T}} \left| m^{(k)}(z)\right|^2 \frac{1}{N^k} \sum_{\substack{w \\ w^{N^k}=z^{N^k}}} |\xi(w)|^2 f(z) \frac{|dz|}{2\pi},
\end{aligned}$$

where the last equality follows from the general formula

$$\int_{\mathbb{T}} g(z) \frac{1}{N^k} \sum_{\substack{v \\ v^{N^k}=z}} h(v) \frac{|dz|}{2\pi} = \int_{\mathbb{T}} g\left(z^{N^k}\right) h(z) \frac{|dz|}{2\pi}. \tag{3.24}$$

We conclude that

$$\int_{\mathbb{T}} f(z)\,|\xi(z)|^2 \frac{|dz|}{2\pi} = \int_{\mathbb{T}} f(z) \left| m^{(k)}(z)\right|^2 \frac{1}{N^k} \sum_{\substack{w \\ w^{N^k}=z^{N^k}}} |\xi(w)|^2 \frac{|dz|}{2\pi}. \tag{3.25}$$

As this equality is valid for all $f \in L^{\infty}(\mathbb{T})$, we conclude that

$$|\xi(z)|^2 = \left|m^{(k)}(z)\right|^2 \frac{1}{N^k} \sum_{\substack{w \\ w^{N^k}=z^{N^k}}} |\xi(w)|^2 \tag{3.26}$$

for almost all $z \in \mathbb{T}$, $k = 1, 2, \dots$.

Now let R_1 be the root mean operator on the measurable functions on $\mathbb{T}$ defined by putting $m = 1$ in (3.15), i.e.,

$$(R_1 \xi)(z) = \frac{1}{N} \sum_{\substack{w \\ w^N=z}} \xi(w). \tag{3.27}$$

If $\varphi \in L^1(\mathbb{T})$, it follows by approximating φ by functions in $C(\mathbb{T})$ that

$$\lim_{k\to\infty} \left\| R_1^k(\varphi) - \left(\int_{\mathbb{T}} \varphi(z) \frac{|dz|}{2\pi} \right) \mathbb{1} \right\|_1 = 0, \tag{3.28}$$

and hence the sequence $R_1^k(\varphi)$ converges to the constant function $\int_{\mathbb{T}} \varphi(z) \frac{|dz|}{2\pi}$ in measure, i.e.,

$$\lim_{k\to\infty} \mu \left\{ z \in \mathbb{T} \;\middle|\; \left| R_1^k(\varphi)(z) - \int_{\mathbb{T}} \varphi(\eta) \frac{|d\eta|}{2\pi} \right| > \varepsilon \right\} = 0$$

for all $\varepsilon > 0$.

But repeating the proof of Birkhoff's mean ergodic theorem [CFS82, Wal82], one can show the stronger conclusion that $R_1^k(\varphi)$ converges almost everywhere to a function which is invariant under all N-adic rotations, and therefore under all rotations, i.e., $R_1^k(\varphi)$ converges almost everywhere to the constant $\int_{\mathbb{T}} \varphi(z) \frac{|dz|}{2\pi}$. But (3.26) says that

$$|\xi(z)|^2 = \left|m^{(k)}(z)\right|^2 R_1^k\left(|\xi|^2\right)\left(z^{N^k}\right), \tag{3.29}$$

and $R_1^k\left(|\xi|^2\right)\left(z^{N^k}\right) \to \|\xi\|_2^2 = 1$ for almost all z by the remarks above, and hence we have proved (3.21):

$$\lim_{k\to\infty} \left|m^{(k)}(z)\right|^2 = |\xi(z)|^2$$

for almost all z. In particular the limit to the left exists for almost all z. Put

$$m_{\infty}(z) = \lim_{k\to\infty} \left|m^{(k)}(z)\right|. \tag{3.30}$$

One consequence of (3.21) is that, if $\xi \in P_U L^2(\mathbb{T})$, then

$$|\xi(z)| = \|\xi\|_2 \, m_{\infty}(z) \tag{3.31}$$

for almost all z; and this immediately implies that the space $P_U L^2(\mathbb{T})$ is one-dimensional, establishing the first statement of Theorem 3.1.

Now, from the relation

$$m^{(k+1)}(z) = m(z)\, m^{(k)}\left(z^N\right) \tag{3.32}$$

and (3.30) we deduce

$$m_{\infty}(z) = |m(z)|\, m_{\infty}\left(z^N\right). \tag{3.33}$$

But, using this and (3.6), we further deduce that

$$\begin{aligned}\sum_{\substack{w\\ w^N=z}} m_\infty(w)^2 &= \sum_{\substack{w\\ w^N=z}} |m(w)|^2 m_\infty(z)^2\\ &= N m_\infty(z)^2,\end{aligned}$$

so

$$\begin{aligned}m_\infty(z)^2 &= \frac{1}{N}\sum_{\substack{w\\ w^N=z}} m_\infty(w)^2\\ &= R_1\left(m_\infty^2\right)(z).\end{aligned}$$

Iterating this, we obtain

$$m_\infty(z)^2 = R_1^k\left(m_\infty^2\right)(z)$$

for $k = 1, 2, 3$; and, letting $k \to \infty$,

$$m_\infty(z)^2 = \int_{\mathbb{T}} m_\infty(w)^2 \frac{|dw|}{2\pi}.$$

Thus $m_\infty(z)$ is a positive constant, and re-inserting this in (3.33) gives

$$|m(z)| = 1$$

for almost all z. Thus from (3.30),

$$m_\infty(z) = 1$$

for almost all z, and then from (3.21),

$$|\xi(z)| = 1$$

for almost all z. This ends the proof of (3.22) and thus of Lemma 3.2. □

Proof of Theorem 3.1. We already commented in connection with (3.31) that if $P_U \neq 0$, then P_U is one-dimensional, and if $P_U \neq 0$, then (3.13) follows from (3.22). But if $P_U L^2(\mathbb{T}) = \mathbb{C}\xi$ with $\|\xi\|_2 = 1$, it follows from unitarity of $S_m P_U = P_U S_m$ that ξ must be an eigenvector of S_m with eigenvalue λ of modulus one, $S_m\xi = \lambda\xi$, or

$$m(z)\,\xi\left(z^N\right) = \lambda\xi(z),$$

which is (3.14).

Conversely, if (3.13) and (3.14) are fulfilled, it is obvious that $P_U \neq 0$, since $\xi \in P_U L^2(\mathbb{T})$ (it suffices instead of (3.13) and (3.14) merely to assume that S_m has an eigenvector with eigenvalues of modulus 1). □

4. The Wold decomposition of operators S_C on $L^2(\mathbb{T};\mathbb{C}^n)$ of the form $(S_C\xi)(z) = C(z)\,\xi\left(z^N\right)$

In this section we will consider a situation which is more general in some respects, and more special in other respects, than in Section 3. Let $L^2(\mathbb{T};\mathbb{C}^n) \cong L^2(\mathbb{T}) \otimes \mathbb{C}^n$ be the Hilbert space of L^2-functions on $\mathbb{T}$ with values in the Hilbert space $\mathbb{C}^n$. Let $C : \mathbb{T} \to M_n = \mathcal{B}(\mathbb{C}^n)$ be a measurable bounded function, and define an operator $S_C \in \mathcal{B}\left(L^2(\mathbb{T};\mathbb{C}^n)\right)$ by

$$(S_C\xi)(z) = C(z)\,\xi\left(z^N\right). \tag{4.1}$$

One verifies as in Section 3 that

$$(S_C^*\xi)(z) = \frac{1}{N}\sum_{\substack{w \\ w^N = z}} C(w)^* \xi(w) \tag{4.2}$$

and hence

$$(S_C^* S_C \xi)(z) = \frac{1}{N}\sum_{\substack{w \\ w^N = z}} C(w)^* C(w)\xi(z). \tag{4.3}$$

Thus S_C is an isometry if and only if

$$\frac{1}{N}\sum_{\substack{w \\ w^N = z}} C(w)^* C(w) = \mathbb{1}_n \tag{4.4}$$

for almost all $z \in \mathbb{T}$. So far everything generalizes Section 3, but in order to prove an analogue of Theorem 3.1 we assume a condition which is a bit stronger, namely that each $C(z)$ is unitary,

$$C(z)^* C(z) = \mathbb{1}_n \tag{4.5}$$

for almost every $z \in \mathbb{T}$. Define as before

$$E_k = S_C^k S_C^{*\,k} \tag{4.6}$$

and let

$$P_U = \operatorname*{s-lim}_{k\to\infty} E_k \tag{4.7}$$

be the projection onto the subspace corresponding to the unitary part of the Wold decomposition of S_C. Again one verifies

$$(E_k\xi)(z) = C^{(k)}(z)\frac{1}{N^k}\sum_{\substack{w \\ w^{N^k} = z^{N^k}}} C^{(k)}(w)^* \xi(w) \tag{4.8}$$

where

$$C^{(k)}(z) = C(z)\,C\left(z^N\right)\cdots C\left(z^{N^{k-1}}\right). \tag{4.9}$$

The analogue of Theorem 3.1 is now

Theorem 4.1. *Assume that S_C is defined by* (4.1) *and assume that the unitarity condition* (4.5) *is satisfied. Then the projection P_U corresponding to the unitary part of the Wold decomposition is at most n-dimensional. If $\dim P_U = m \le n$, the range of P_U can be characterized as follows: there is a projection $P_0 \in M_n$ of dimension m, and a measurable function $\Delta : \mathbb{T} \to M_n$, such that*

$$\Delta(z)^* \Delta(z) = P_0 \tag{4.10}$$

for all $z \in \mathbb{T}$, i.e., $\Delta(z)$ is a partial isometry with initial projection P_0, and a function $\xi \in L^2(\mathbb{T};\mathbb{C}^n)$ is in the range of P_U if and only if there is a vector $v \in P_0\mathbb{C}^n$ such that

$$\xi(z) = \Delta(z)\,v \tag{4.11}$$

for almost all z. Furthermore, there is a partial unitary $U_0 \in M_n$ with

$$U_0 U_0^* = U_0^* U_0 = P_0 \tag{4.12}$$

such that

$$C(z)\,\Delta\left(z^N\right) = \Delta(z)\,U_0. \tag{4.13}$$

Here P_0 is the unique maximal projection with the property that there exist $\Delta(\,\cdot\,)$ and U_0 satisfying (4.10), (4.12), *and* (4.13); *and then U_0 is uniquely determined, and $\Delta(\,\cdot\,)$ is uniquely determined up to a phase factor.*

Proof. Let $\xi \in P_U(\mathcal{H})$. For all k we have

$$E_k \xi = \xi, \tag{4.14}$$

and it follows from (4.8) that

$$C^{(k)}(z)^*\,\xi(z) = \frac{1}{N^k} \sum_{\substack{w \\ w^{N^k} = z^{N^k}}} C^{(k)}(w)^*\,\xi(w) \tag{4.15}$$

for almost all $z \in \mathbb{T}$. But replacing the z to the left with any $\eta \in \mathbb{T}$ with $\eta^{N^k} = z^{N^k}$, we see that any of the vectors $C^{(k)}(\eta)^*\,\xi(\eta)$ is a convex combination of all the vectors of this form with equal weight, and it follows that

$$C^{(k)}(z)^*\,\xi(z) = C^{(k)}(w)^*\,\xi(w) \tag{4.16}$$

whenever $z^{N^k} = w^{N^k}$. At this point we use the unitarity of $C^{(k)}(w)^*$ to deduce

$$\|\xi(z)\| = \|\xi(w)\| \tag{4.17}$$

whenever $z^{N^k} = w^{N^k}$, and letting $k \to \infty$ and using Luzin's theorem (for any $\varepsilon > 0$ there is a closed subset $F \subset \mathbb{T}$ such that $\mu(\mathbb{T} - F) < \varepsilon$ and $z \to \|\xi(z)\|$ is continuous on F, where μ is Haar measure) we deduce that $\|\xi(z)\|$ is equal to a constant for almost all z. Now, if $\xi, \eta \in P_U(\mathcal{H})$, then all linear combinations of ξ and η are in $P_U(\mathcal{H})$, and it follows from the polarization identity

$$\langle \xi(z) \,|\, \eta(z) \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \left\| \xi(z) + i^k \eta(z) \right\| \tag{4.18}$$

that $z \to \langle \xi(z) \,|\, \eta(z) \rangle$ is equal to a constant almost everywhere. It follows that $P_U(\mathcal{H})$ can at most be n-dimensional, and modifying the representatives $\xi(z)$ on a set of measure zero, we may assume that

$$z \to \langle \xi(z) \,|\, \eta(z) \rangle \tag{4.19}$$

is a constant for any two $\xi, \eta \in P_U(\mathcal{H})$. But then, if P_0 is the projection onto the set of $\xi(1)$, $\xi \in P_U(\mathcal{H})$, we may for each $v \in P_0\mathbb{C}^n$ find a ξ with $\xi(1) = v$, and define

$$\Delta(z)\,v = \Delta(z)\,\xi(1) = \xi(z)\,. \tag{4.20}$$

Because of (4.19), each $\Delta(z)$ is a partial isometry with initial projection P_0, and the statements around (4.10) and (4.11) in the theorem are proved. Furthermore, we have defined a unitary operator $V : P_0(\mathbb{C}^n) \to P_U(\mathcal{H})$ by

$$(Vv)(z) = \Delta(z)\,v.$$

But as $P_U S_C P_U$ is unitary on $P_U\mathcal{H}$, we have that

$$U_0 = V^* P_U S_C P_U V = V^* S_C V$$

is unitary on $P_0\mathbb{C}^n$. But as

$$VU_0v = S_C Vv$$

for $v \in P_0\mathbb{C}^n$ we have

$$\Delta(z) U_0 v = C(z) \Delta\left(z^N\right) v$$

and (4.13) follows. Conversely, it is easy to check from (4.13) that $\xi(z) = \Delta(z) v$ is in the range of each E_k. This ends the proof of Theorem 4.1. □

5. The Wold decomposition of operators T on $L^2(\mathbb{T})$ of the form $(T\xi)(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} m_k(z) \xi\left(\rho^k z^N\right)$

Here $\rho = \rho_N = e^{\frac{2\pi i}{N}}$, and the coefficient functions $m_k \in L^\infty(\mathbb{T})$ satisfy the unitarity condition (1.11), i.e.,

$$C(z) := \frac{1}{\sqrt{N}} \begin{pmatrix} m_0(z) & m_1(z) & \dots & m_{N-1}(z) \\ m_0(\rho z) & m_1(\rho z) & \dots & m_{N-1}(\rho z) \\ \vdots & \vdots & \ddots & \vdots \\ m_0(\rho^{N-1}z) & m_1(\rho^{N-1}z) & \dots & m_{N-1}(\rho^{N-1}z) \end{pmatrix} \tag{5.1}$$

is unitary for almost every $z \in \mathbb{C}$. This condition implies that the operator T defined by

$$(T\xi)(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} m_k(z) \xi\left(\rho^k z^N\right) \tag{5.2}$$

is an isometry, since T has the form

$$T = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k U^k, \tag{5.3}$$

where $S_0, \dots, S_{N-1}$ is the representation of $\mathcal{O}_N$ given by (1.16), and U is the unitary operator on $L^2(\mathbb{T})$ defined by

$$(U\xi)(z) = \xi(\rho z).$$

Let S_C be the isometry on $L^2\left(\mathbb{T}; \mathbb{C}^N\right)$ defined by (4.1),

$$(S_C\xi)(z) = C(z) \xi\left(z^N\right). \tag{5.4}$$

We now verify that S_C is a dilation of T. Define an isometric embedding $V : L^2(\mathbb{T}) \to L^2\left(\mathbb{T}; \mathbb{C}^N\right)$ by

$$(V\xi)(z) = \frac{1}{\sqrt{N}} \begin{pmatrix} \xi(z) \\ \xi(\rho z) \\ \vdots \\ \xi(\rho^{N-1}z) \end{pmatrix}. \tag{5.5}$$

The dilation property is then given by

$$S_C V = VT, \tag{5.6}$$

which is easily verified from (5.1), (5.2), (5.4), and (5.5).

If S is a general isometry, let $\mathcal{H}_U(S)$ denote the subspace of the Hilbert space corresponding to the unitary part of the Wold decomposition of S, i.e.,

$$\mathcal{H}_U(S) = \bigcap_k S^k S^{*\,k} \mathcal{H}. \tag{5.7}$$

It turns out that the unitary subspaces of T and of its dilation S_C are the same!

Proposition 5.1. *With the assumptions and notation above,*

$$\mathcal{H}_U(S_C) = V(\mathcal{H}_U(T)). \tag{5.8}$$

Proof. Since S_C is a dilation of T in the sense of (5.6), it is clear that

$$V(\mathcal{H}_U(T)) \subset \mathcal{H}_U(S), \tag{5.9}$$

and to prove the reverse inclusion it suffices to show that any $\eta \in \mathcal{H}_U(S)$ is in the range of V, i.e., that there is a $\xi \in L^2(\mathbb{T})$ such that

$$\eta(z) = \begin{pmatrix} \xi(z) \\ \xi(\rho z) \\ \xi(\rho^2 z) \\ \vdots \\ \xi(\rho^{N-1} z) \end{pmatrix}. \tag{5.10}$$

But by Theorem 4.1, η has the form

$$\eta(z) = \Delta(z)\, v$$

for a suitable $v \in \mathbb{C}^N$, and by linearity we may assume that v is an eigenvector of the partial unitary matrix U_0, i.e.,

$$U_0 v = \lambda v,$$

where $\lambda \in \mathbb{T}$. We then obtain from (4.13) that

$$C(z)\,\eta\left(z^N\right) = \lambda \eta(z).$$

If

$$\eta(z) = \begin{pmatrix} \xi_0(z) \\ \vdots \\ \xi_{N-1}(z) \end{pmatrix},$$

we thus obtain from (5.1) that

$$\begin{pmatrix} \xi_0(z) \\ \xi_1(z) \\ \vdots \\ \xi_{N-1}(z) \end{pmatrix} = \frac{\bar{\lambda}}{\sqrt{N}} \begin{pmatrix} m_0(z) & m_1(z) & \dots & m_{N-1}(z) \\ m_0(\rho z) & m_1(\rho z) & \dots & m_{N-1}(\rho z) \\ \vdots & \vdots & \ddots & \vdots \\ m_0(\rho^{N-1} z) & m_1(\rho^{N-1} z) & \dots & m_{N-1}(\rho^{N-1} z) \end{pmatrix} \begin{pmatrix} \xi_0(z^N) \\ \xi_1(z^N) \\ \vdots \\ \xi_{N-1}(z^N) \end{pmatrix},$$

and hence, using $\rho^{kN}=1$,

$$\begin{aligned}\xi_k(z) &= \bar{\lambda}\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} m_j\left(\rho^k z\right)\xi_j\left(z^N\right)\\ &= \xi_0\left(\rho^k z\right).\end{aligned}$$

Thus η has the special form (5.10), and we have established the reverse inclusion of (5.9), and thus Proposition 5.1. ☐

Let us summarize the results of this section and the previous one.

Corollary 5.2. *The subspace corresponding to the unitary part of the Wold decomposition of the operator T defined by* (5.2) *has dimension $n \le N$. Furthermore, there exists a projection $P_0 \in M_N$ of dimension n, and measurable functions $d_0(z), \dots, d_{N-1}(z)$ from $\mathbb{T}$ into $\mathbb{C}$ such that*

$$\sum_{k\in\mathbb{Z}_N} \bar{d}_i\left(\rho^k z\right) d_j\left(\rho^k z\right) = \left(P_0\right)_{ij} \tag{5.11}$$

for $i, j \in \mathbb{Z}_N$ and almost all z, such that $\xi \in \mathcal{H}_U(T)$ if and only if there are scalars $v_0, \dots, v_{N-1}$ with

$$\xi(z) = \sum_k d_k(z)\, v_k. \tag{5.12}$$

If

$$\Delta(z) = \begin{pmatrix} d_0(z) & d_1(z) & \dots & d_{N-1}(z) \\ d_0(\rho z) & d_1(\rho z) & \dots & d_{N-1}(\rho z) \\ \vdots & \vdots & \ddots & \vdots \\ d_0(\rho^{N-1}z) & d_1(\rho^{N-1}z) & \dots & d_{N-1}(\rho^{N-1}z) \end{pmatrix} \tag{5.13}$$

then there exists a partial unitary $U_0 \in M_N$ with

$$U_0 U_0^* = U_0^* U_0 = P_0 \tag{5.14}$$

such that

$$C(z)\,\Delta\left(z^N\right) = \Delta(z)\, U_0 \tag{5.15}$$

for almost all $z \in \mathbb{T}$. In particular, if $v = \begin{pmatrix} v_0 \\ \vdots \\ v_{N-1}\end{pmatrix}$ is taken to be an eigenvector for U_0 with eigenvalue $\lambda \in \mathbb{T}$, then $\xi(z) = \sum_k d_k(z)\, v_k$ is a Haar vector in the sense

$$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} m_k(z)\,\xi\left(\rho^k z^N\right) = \bar{\lambda}\xi(z). \tag{5.16}$$

Proof. Let U_0, P_0, $\Delta(z)$ be the objects constructed in Theorem 4.1 from the matrix $C(z)$. If we write $\Delta(z)$ as a column vector of row vectors,

$$\Delta(z) = \begin{pmatrix} \Delta_0(z) \\ \vdots \\ \Delta_{N-1}(z) \end{pmatrix}, \tag{5.17}$$

it follows from

$$C(z) = \begin{pmatrix} C_0(z) \\ C_0(\rho z) \\ \vdots \\ C_0(\rho^{N-1}z) \end{pmatrix}, \tag{5.18}$$

where $C_0(z)$ is the first row of $C(z)$, and (4.13), that

$$\begin{pmatrix} \Delta_0(z) \\ \vdots \\ \Delta_{N-1}(z) \end{pmatrix} = \begin{pmatrix} C_0(z) \\ C_0(\rho z) \\ \vdots \\ C_0(\rho^{N-1}z) \end{pmatrix} \begin{pmatrix} \Delta_0(z^N) \\ \vdots \\ \Delta_{N-1}(z^N) \end{pmatrix} U_0^*, \tag{5.19}$$

and hence

$$\Delta_k(z) = \Delta_0\left(\rho^k z\right) \tag{5.20}$$

for $k \in \mathbb{Z}_N$. It follows that $\Delta(z)$ has the form in (5.13), and then (5.11), (5.12), (5.14), and (5.15) are transcriptions of (4.10), (4.11), (4.12), and (4.13), respectively. $\square$

6. Characterizations of Cuntz algebra representations with S_0 a shift and realizations of these representations on a Hardy space

Recall that an isometry S on a Hilbert space $\mathcal{H}$ is called (by us) a shift iff $\bigcap_{n=1}^{\infty} S^n\mathcal{H} = \{0\}$. Then putting $\mathcal{K} = \mathcal{H} \ominus S\mathcal{H}$, letting $\xi_{0,j}$ be an orthonormal basis for $\mathcal{K}$ and putting $\xi_{i,j} = S^i\xi_{0,j}$, $i \in \mathbb{N} \cup \{0\}$, $\{\xi_{i,j}\}$ is an orthonormal basis for $\mathcal{H}$. Let $H_+^2(\mathbb{T})$ be the Hardy subspace of $L^2(\mathbb{T})$, i.e., the closed linear span of the orthonormal set of functions $z \mapsto z^n$, $n = 1, 2, 3, \dots$, and define a unitary operator

$$V : \mathcal{H} \to H_+^2(\mathbb{T}) \otimes \mathcal{K} = \mathcal{H}_+(\mathcal{K}) \tag{6.1}$$

by

$$V\xi_{i,j} = z^{i+1} \otimes \xi_{0,j}. \tag{6.2}$$

Viewing the elements in $\mathcal{H}_+(\mathcal{K})$ as functions from $\mathbb{T}$ into $\mathcal{K}$,

$$\xi \in \mathcal{H}_+(\mathcal{K}) \Longleftrightarrow \xi(z) = \sum_{n=1}^{\infty} \xi_n z^n,$$

where $\xi_n \in \mathcal{K}$ and $\|\xi\|^2 = \sum_{n=1}^{\infty} \|\xi_n\|^2$, $S_0^+ = VS_0V^*$ is nothing but the operator of multiplication by z:

$$\left(S_0^+\xi\right)(z) = z\xi(z). \tag{6.3}$$

We will now generalize this description of a shift to a representation $s_i \to S_i$ of the Cuntz algebra $\mathcal{O}_N$ on $\mathcal{H}$ such that S_0 is a shift, when $N = 2, 3, \dots$.

Lemma 6.1. *There is a 1–1 correspondence between representations $s_i \to S_i$ of $\mathcal{O}_N$ on $\mathcal{H}$ such that S_0 is a shift, and representations of $\mathcal{O}_\infty$ on $\mathcal{H}$ such that the sum of the ranges of the isometries is $\mathbb{1}$. If the representatives of the generators of $\mathcal{O}_\infty$ are denoted by $T_k^{(\infty,j)}$, where $j = 1, \dots, N-1$, $k = 1, 2, 3, \dots$, so that*

$$T_{k_1}^{(\infty,j_1)*} T_{k_2}^{(\infty,j_2)} = \delta_{j_1 j_2}\delta_{k_1 k_2}\mathbb{1} \tag{6.4}$$

and

$$\sum_{j=1}^{N-1}\sum_{k=1}^{\infty} T_k^{(\infty,j)}\, T_k^{(\infty,j)*} = \mathbb{1}, \tag{6.5}$$

then the 1–1 *correspondence is given by*

$$T_k^{(\infty,j)} = S_0^{k-1}S_j, \quad j=1,\dots,N-1, \quad k=1,2,\dots, \tag{6.6}$$

and by

$$S_0 = \sum_{k=1}^{\infty}\sum_{j=1}^{N-1} T_{k+1}^{(\infty,j)}\, T_k^{(\infty,j)*} \tag{6.7}$$

and

$$S_j = T_1^{(\infty,j)}, \quad j=1,\dots,N-1, \tag{6.8}$$

where all infinite sums converge in the strong operator topology.

Proof. If $S_0,\dots,S_{N-1}$ is a representation of $\mathcal{O}_N$ on $\mathcal{H}$ with S_0 a shift, define $T_k^{(\infty,j)}$ by (6.6). One uses the Cuntz relations

$$S_i^* S_j = \delta_{ij}\mathbb{1} \tag{6.9}$$

to verify (6.4). The other Cuntz relation,

$$\sum_j S_j S_j^* = \mathbb{1}, \tag{6.10}$$

implies

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{j=1}^{N-1} T_k^{(\infty,j)}\, T_k^{(\infty,j)*} &= \sum_{k=1}^{\infty}\sum_{j=1}^{N-1} S_0^{k-1} S_j S_j^* S_0^{*\,k-1}\\ &= \sum_{k=1}^{\infty} S_0^{k-1}\left(\mathbb{1} - S_0S_0^*\right) S_0^{*\,k-1}\\ &= \mathbb{1} - \lim_{k\to\infty} S_0^k S_0^{*\,k},\end{aligned}$$

but the last limit is zero since S_0 is a shift, and (6.5) follows. Furthermore, (6.8) is immediate from (6.6), while

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{j=1}^{N-1} T_{k+1}^{(\infty,j)}\, T_k^{(\infty,j)*} &= \sum_{k=1}^{\infty}\sum_{j=1}^{N-1} S_0^{k} S_j S_j^* S_0^{*\,k-1}\\ &= \sum_{k=1}^{\infty} S_0^{k}\left(\mathbb{1} - S_0S_0^*\right) S_0^{*\,k-1}\\ &= S_0\left(\mathbb{1} - \lim_{k\to\infty} S_0^k S_0^{*\,k}\right)\\ &= S_0,\end{aligned}$$

so (6.7) is verified.

Conversely, if $T_k^{(\infty,j)}$ are given satisfying (6.4) and (6.5), one verifies that S_0 and S_j, $j=1,\dots,N-1$, satisfy the Cuntz relations (6.9) and (6.10), that (6.6) is valid, and that $\lim_{k\to\infty} S_0^k S_0^{*\,k} = 0$, i.e., S_0 is a shift. ∎

We will now use this Lemma to construct the announced Hardy-space structure on $\mathcal{H}$.

Theorem 6.2. *Let $s_i \rightarrow S_i$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{K}$ such that S_0 is a shift. Then there exists a unitary operator*

$$V : \mathcal{H}_+ \left(\bigoplus_{j=1}^{N-1} \mathcal{K} \right) \longrightarrow \mathcal{K} \tag{6.11}$$

such that if $S_j^+ = V^* S_j V$, then

$$S_0^+ = M_z = \textit{multiplication by } z \tag{6.12}$$

and

$$S_j^+ \psi = z \left(\left(\bigoplus_{i=1}^{j-1} 0 \right) \oplus V\psi \oplus \left(\bigoplus_{i=j+1}^{N-1} 0 \right) \right) \tag{6.13}$$

for $j = 1, \dots, N-1$.

Proof. Define $T_k^{(\infty,j)}$, $j = 1, \dots, N-1$, $k = 1, 2, \dots$, by (6.6), and define $V : \mathcal{H}_+ \left(\bigoplus_{j=1}^{N-1} \mathcal{K} \right) \longrightarrow \mathcal{K}$ by

$$V \left(\sum_{k=1}^{\infty} \left(\bigoplus_{j=1}^{N-1} \psi_k^{(j)} \right) z^k \right) = \sum_{k=1}^{\infty} \sum_{j=1}^{N-1} T_k^{(\infty,j)} \psi_k^{(j)}. \tag{6.14}$$

It follows from (6.4) and (6.5) in Lemma 6.1 that V is indeed unitary and

$$V^* \psi = \sum_{k=1}^{\infty} \left(\bigoplus_{j=1}^{N-1} T_k^{(\infty,j)*} \psi \right) z^k. \tag{6.15}$$

Thus, if $\psi(z) = \sum_{k=1}^{\infty} \left(\bigoplus_{j=1}^{N-1} \psi_k^{(j)} \right) z^k$ is in $\mathcal{H}_+ \left(\bigoplus_{j=1}^{N-1} \mathcal{K} \right)$, one checks

$$\begin{aligned}
\left(S_0^+ \psi \right)(z) &= \left(V^* S_0 V \psi \right)(z) \\
&= \left(V^* \left(S_0 \left(\sum_{k=1}^{\infty} \sum_{j=1}^{N-1} T_k^{(\infty,j)} \psi_k^{(j)} \right) \right) \right)(z) \\
&= \left(V^* \left(\sum_{k=1}^{\infty} \sum_{j=1}^{N-1} T_{k+1}^{(\infty,j)} \psi_k^{(j)} \right) \right)(z) \\
&= \sum_{k=2}^{\infty} \left(\bigoplus_{j=1}^{N-1} \psi_{k-1}^{(j)} \right) z^k = z\psi(z),
\end{aligned}$$

where we used

$$S_0 T_k^{(\infty,j)} = T_{k+1}^{(\infty,j)},$$

which follows from (6.6). Thus (6.12) is valid. For $j = 1, 2, \ldots, N-1$ we use (6.8) and (6.15) to calculate

$$\begin{aligned}\left(S_j^{+}\psi\right)(z) = \left(V^{*}S_jV\psi\right)(z) &= \left(V^{*}\left(S_j\left(\sum_{k=1}^{\infty}\sum_{i=1}^{N-1}T_k^{(\infty,i)}\psi_k^{(i)}\right)\right)\right)(z)\\ &= z\left(\left(\bigoplus_{i=1}^{j-1}0\right)\oplus V\psi\oplus\left(\bigoplus_{i=j+1}^{N-1}0\right)\right),\end{aligned}$$

which is (6.13). ∎

7. Characterization of representations π of $\mathcal{O}_N$ with $\pi(\mathcal{D}_N) \subset M_{L^\infty(\mathbb{T})}$

Recall that UHF_N is the C^*-subalgebra of $\mathcal{O}_N$ which is the closed linear span of $s_I s_J^*$ with $|I| = |J|$, and $\mathcal{D}_N$ is the canonical maximal abelian subalgebra of UHF_N, i.e., $\mathcal{D}_N$ is the closed linear span of operators $s_I s_I^*$. Thus, if $\mathrm{UHF}_N \cong \bigotimes_{n=1}^{\infty} M_N$, then $\mathcal{D}_N \cong \bigotimes_{n=1}^{\infty} \mathbb{C}^N$.

Theorem 7.1. *Consider a representation π of $\mathcal{O}_N$ on $L^2(\mathbb{T})$ of the form* (1.16),

$$\left(S_i\xi\right)(z) = m_i(z)\,\xi\left(z^N\right),$$

where the functions m_i satisfy the appropriate form of the unitarity condition (1.11). *Let $M_{L^\infty(\mathbb{T})}$ be the image of $L^\infty(\mathbb{T})$ acting as multiplication operators on $L^2(\mathbb{T})$. The following conditions are equivalent:*

(7.1) $\quad \pi(\mathcal{D}_N)'' \subset M_{L^\infty(\mathbb{T})}$;

(7.2) $\quad \pi(\mathcal{D}_N)'' = M_{L^\infty(\mathbb{T})}$;

(7.3) $\quad m_i(z) = \sqrt{N}\chi_{A_i}(z)\,u(z)$,

where u is a measurable function $\mathbb{T} \to \mathbb{T}$, and $A_0, \ldots, A_{N-1}$ are N measurable subsets of $\mathbb{T}$ with the property that if $\rho = e^{\frac{2\pi i}{N}}$, then, for almost all $z \in \mathbb{T}$, the N equidistant points $z, \rho z, \rho^2 z, \ldots, \rho^{N-1}z$ lie with one in each of the N sets $A_0, \ldots, A_{N-1}$ (i.e., $A_0, \ldots, A_{N-1}$ form a partition of $\mathbb{T}$ up to null sets, and, for each k, the N sets $A_k, \rho A_k, \ldots, \rho^{N-1}A_k$ form a partition of $\mathbb{T}$). Any m_i of this form does indeed define a representation of $\mathcal{O}_N$.

Proof. (7.2)⇒(7.1) is trivial, and we prove (7.1)⇒(7.3)⇒(7.2).

Ad (7.1)⇒(7.3): Assume (7.1). From (1.16)–(1.17), we have

$$\left(S_iS_i^*\xi\right)(z) = \frac{1}{N}m_i(z)\sum_{\substack{w\\ w^N=z^N}}\bar{m}_i(w)\,\xi(w). \tag{7.4}$$

But in order that $S_iS_i^*$ be a multiplication operator, we must thus require that

$$m_i(z)\,\bar{m}_i(w) = 0$$

almost everywhere, whenever $w^N = z^N$ and $w \neq z$. If A_i is the support of m_i, it follows that the N sets A_i, $\rho A_i, \ldots, \rho^{N-1}A_i$ are disjoint (up to null sets). Thus, for given $z \in \mathbb{T}$ and $i \in \mathbb{Z}_N$, there is at most one j such that $\rho^j z \in A_i$. But, by the unitarity condition (1.11), for given i and z,

$$\sum_j\left|m_i\left(\rho^j z\right)\right|^2 = N,$$

and hence there must exist a j with $\rho^j z \in A_i$. We have thus proved that the points $z, \rho z, \rho^2 z, \dots, \rho^{N-1} z$ lie one each in the sets $A_0, A_1, \dots, A_{N-1}$, so these sets form a partition with the stated properties. But then necessarily the functions m_i must have the form

$$m_i(z) = \sqrt{N}\chi_{A_i}(z)\,u(z),$$

where $u : \mathbb{T} \to \mathbb{C}$ is a measurable function. But, by unitarity again,

$$1 = \frac{1}{N}\sum_i |m_i(z)|^2 = |u(z)|^2,$$

so that u actually maps into the circle $\mathbb{T}$. Thus (7.3) is valid, and we just comment at this point that if $m_i(z)$ is given by (7.3), the matrix (1.11) is a permutation matrix for any given x, and thus the unitarity condition is fulfilled from the conditions in (7.3) and the last statement of the theorem follows.

Ad (7.3)⇒(7.2): Assume (7.3). Using (1.16)–(1.17), and putting $\rho = \rho_n = e^{\frac{2\pi i}{N^n}}$, one has for $I = (i_1, \dots, i_n)$:

$$\begin{aligned} (7.5)\quad & (S_I S_I^* \xi)(z) \\ &= m_{i_1}(z)\, m_{i_2}\left(z^N\right)\cdots m_{i_n}\left(z^{N^{n-1}}\right)\frac{1}{N^n} \\ &\quad\cdot \sum_{k=0}^{N^n-1} \bar{m}_{i_n}\left(\rho^{kN^{n-1}} z^{N^{n-1}}\right)\bar{m}_{i_{n-1}}\left(\rho^{kN^{n-2}} z^{N^{n-2}}\right)\cdots \bar{m}_{i_1}\left(\rho^k z\right)\xi\left(\rho^k z\right) \\ &= \chi_{A_{i_1}}(z)\,\chi_{A_{i_2}}\left(z^N\right)\cdots\chi_{A_{i_n}}\left(z^{N^{n-1}}\right)\xi(z). \end{aligned}$$

But now defining a coding map $\sigma : \mathbb{T} \to \prod_{k=1}^{\infty}\mathbb{Z}_n$ by $\sigma(z) = (i_1, i_2, i_3, \dots)$, if $z^{N^{n-1}} \in A_{i_n}$, it follows from the properties of A_i that σ is a measure-preserving map from $\mathbb{T}$ into $\prod_{k=1}^{\infty}\mathbb{Z}_n$ when both groups are equipped with normalized Haar measure. Replacing $\mathbb{T}$ by $\prod_{k=1}^{\infty}\mathbb{Z}_n$ by means of this map, the relation (7.5) takes the form

$$(7.6)\qquad (S_I S_I^* \xi)(j_1, j_2, \dots) = \delta_{i_1 j_1}\delta_{i_2 j_2}\cdots\delta_{i_n j_n}\xi(j_1, j_2, \dots),$$

and it is clear from this relation that the von Neumann algebra generated by $\pi(\mathcal{D}_N)$ is exactly $L^\infty\left(\bigtimes_{n=1}^{\infty}\mathbb{Z}_N\right)$. This establishes (7.3)⇒(7.2), and Theorem 7.1 is proved. □

From the last part of the above proof, we also have the following

Corollary 7.2. *If π is a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$ satisfying the equivalent conditions* (7.1)–(7.3) *in Theorem* 7.1, *then there is a unitary operator $U : \mathcal{H} \to L^2\left(\bigtimes_{n=1}^{\infty}\mathbb{Z}_n\right)$ and a measurable function $v : \bigtimes_{n=1}^{\infty}\mathbb{Z}_n \to \mathbb{T}$ such that*

$$(7.7)\qquad (US_iU^*\xi)(j_1, j_2, \dots) = \sqrt{N}\delta_{i j_1} v(j_1, j_2, \dots)\,\xi(j_2, j_3, \dots)$$

for all $(j_1, j_2, \dots) \in \bigtimes_{n=1}^{\infty}\mathbb{Z}_n$ and all $\xi \in L^2\left(\bigtimes_{n=1}^{\infty}\mathbb{Z}_n\right)$.

Let us finally remark that the representations described in Corollary 7.2 are all irreducible, even in restriction to UHF_N, by [BrJo96a, Proposition 7.1].

8. Classification of representations with $\pi(\mathcal{D}_N)'' \subset M_{L^\infty(\mathbb{T})}$ up to unitary equivalence

We will now make a further study of the representations of $\mathcal{O}_N$ introduced in Section 7. By Corollary 7.2, these act on $L^2\left(\prod_1^\infty \mathbb{Z}_N\right)$ and are labeled by measurable functions

$$u : \prod_1^\infty \mathbb{Z}_N \to \mathbb{T}. \tag{8.1}$$

In the case that u depends only on a finite number of the coordinates in $\prod_1^\infty \mathbb{Z}_N$, these representations were also considered in Section 7 in [BrJo96a]. The formulae (1.12)–(1.13) now take the form

$$(S_i\xi)(x_1, x_2, \dots) = \sqrt{N}\delta_{x_1 i} u(x_1, x_2, \dots)\,\xi(x_2, x_3, \dots), \tag{8.2}$$

$$(S_i^*\xi)(x_1, x_2, \dots) = \frac{1}{\sqrt{N}}\bar{u}(i, x_1, x_2, \dots)\,\xi(i, x_1, x_2, \dots). \tag{8.3}$$

We define π^u as the representation defined by u. By emulating the proof of irreducibility of π^u from [BrJo96a] we can now establish the following

Proposition 8.1. *Let T be a bounded operator on $L^2\left(\prod_1^\infty \mathbb{Z}_N\right)$, and let $u, u' : \prod_1^\infty \mathbb{Z}_N \to \mathbb{T}$ be measurable functions. Then the following conditions are equivalent.*

(8.4) $T\pi^u(x) = \pi^{u'}(x)\,T$ *for all* $x \in \mathcal{O}_N$.

(8.5) $T = M_f$ *where* $f \in L^\infty\left(\prod_1^\infty \mathbb{Z}_N\right)$ *is a function satisfying*

$$f(x_1, x_2, \dots)\,u(x_1, x_2, \dots) = u'(x_1, x_2, \dots)\,f(x_2, x_3, \dots)$$

for all $(x_1, x_2, \dots) \in \prod_1^\infty \mathbb{Z}_N$.

Remark 8.2. In particular, if $u = u'$, (8.5) entails $f = f \circ \sigma$; and hence f is constant by ergodicity, and this confirms the irreducibility of π^u.

Proof. *Ad* (8.4)⇒(8.5): By (7.6), we have

$$(\pi^u(s_I s_I^*)\,\xi)(x_1, x_2, \dots) = \delta_{i_1 x_1}\delta_{i_2 x_2}\cdots\delta_{i_n x_n}\xi(x_1, x_2, \dots), \tag{8.6}$$

where the right side is independent of u, and hence the intertwining operator T must commute with $M_{L^\infty(\prod_1^\infty \mathbb{Z}_N)}$, and as the latter algebra is maximal abelian there must be an $f \in L^\infty\left(\prod_1^\infty \mathbb{Z}_N\right)$ such that

$$T = M_f.$$

But then

$$(T\pi^u(s_i)\,\xi)(x_1, x_2, \dots) = f(x_1, x_2, \dots)\sqrt{N}\delta_{x_1 i} u(x_1, x_2, \dots)\,\xi(x_2, x_3, \dots)$$

and

$$\left(\pi^{u'}(s_i)\,T\xi\right)(x_1, x_2, \dots) = \sqrt{N}\delta_{x_1 i} u'(x_1, x_2, \dots)\,f(x_2, x_3, \dots)\,\xi(x_2, x_3, \dots).$$

The intertwining (8.4) for $x = s_i$ implies that

$$f(x_1, x_2, \dots)\,u(x_1, x_2, \dots) = u'(x_1, x_2, \dots)\,f(x_2, x_3, \dots).$$

This ends the proof of (8.4)⇒(8.5). For the converse implication, note that the relation in (8.5) and the computation above imply

$$M_f \pi^u (s_i) = \pi^{u'} (s_i) M_f.$$

But as M_f is normal, it follows from Fuglede's theorem (or a direct computation) that

$$M_f \pi^u (s_i^*) = \pi^{u'} (s_i^*) M_f,$$

and hence (8.4) is valid. □

If we view u, u' as functions on $\mathbb{T}$, the result in Proposition 8.1 can be stated in terms of the cohomology theory of Section 2:

Corollary 8.3. *Let $u, u' : \mathbb{T} \to \mathbb{T}$ be measurable functions and π^u, $\pi^{u'}$ be the associated irreducible representations of $\mathcal{O}_N$. Then the following conditions are equivalent.*

(8.7) *π^u and $\pi^{u'}$ are unitarily equivalent.*

(8.8) *The cocycles u, u' cobound, i.e., there exists a measurable function $\Delta : \mathbb{T} \to \mathbb{T}$ such that*

$$\Delta (z) u (z) = u' (z) \Delta \left(z^N\right)$$

for almost all $z \in \mathbb{T}$.

Proof. By Proposition 8.1, π^u and $\pi^{u'}$ are unitarily equivalent if and only if there is a nonzero function $f : \mathbb{T} \to \mathbb{C}$ with

$$f (z) u (z) = u' (z) f \left(z^N\right).$$

But as $|u (z)| = |u' (z)| = 1$ we obtain

$$|f (z)| = \left|f \left(z^N\right)\right|,$$

and by ergodicity of $z \mapsto z^N$, $z \mapsto |f (z)|$ is equal to a constant almost everywhere. Let $\Delta (z) = f (z) / |f (z)|$. Then Δ satisfies (8.8). Conversely, if Δ satisfies (8.8), then M_Δ is an intertwiner between the two representations. □

In conclusion, the unitary equivalence classes of the representations π^u are labeled by the cohomology classes of the cocycles u, which are discussed in Section 2.

9. Computation of the Hardy-space realizations for the examples coming from wavelets

In Section 6 we defined a certain Hardy-space realization of representations of $\mathcal{O}_N$ in the slightly special case that S_0 is a shift. The construction depended on some seemingly arbitrary choices. However, in the case that $N = 2$ and the representation of $\mathcal{O}_2$ comes from a wavelet in $L^2 (\mathbb{R})$ as described in [Jor95], it turns out that these choices are quite canonical. We will describe this in the present section.

Let us first give a short rundown of the multiresolution analysis of wavelets from [Dau92], [MePa93]. The starting point is a function $\varphi \in L^2 (\mathbb{R})$, called the *scaling function* or *father function* with the properties (9.1), (9.3), (9.4a), and (9.4b) below.

(9.1) The set $\{\varphi (\,\cdot\, - k) \mid k \in \mathbb{Z}\}$ is an orthonormal set in $L^2 (\mathbb{R})$.

If we define $\mathcal{V}_0$ as the closed linear span of the functions $\varphi(\,\cdot\, - k)$, it is then clear that $\mathcal{V}_0$ is invariant under $\mathbb{Z}$-translation. Define *scaling* on $L^2(\mathbb{R})$ as the unitary operator U:

$$(U\xi)(x) = 2^{-\frac{1}{2}}\xi(x/2)\,. \tag{9.2}$$

Then we assume

$$U\varphi \in \mathcal{V}_0. \tag{9.3}$$

We define the multiresolution associated to φ as the sequence of subspaces $\mathcal{V}_n = U^n\mathcal{V}_0$, and this sequence is decreasing by (9.3). The final assumptions on φ are

$$\bigwedge_n \mathcal{V}_n = \{0\}\,, \tag{9.4a}$$

$$\bigvee_n \mathcal{V}_n = L^2(\mathbb{R})\,. \tag{9.4b}$$

From φ one now constructs a *wavelet* or *mother function* ψ as follows: first use (9.3) and expand $U\varphi$ in the orthonormal basis $\varphi(\,\cdot\, - k)$:

$$U\varphi = \sum_k a_k\varphi(\,\cdot\, - k)\,. \tag{9.5}$$

Equivalently, using the Fourier transform

$$\hat{\varphi}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ixt}\varphi(x)\,, \tag{9.6}$$

the relation (9.5) takes the form

$$\sqrt{2}\hat{\varphi}(2t) = m_0(t)\,\hat{\varphi}(t)\,, \tag{9.7}$$

where

$$m_0(t) = \sum_k a_k e^{-ikt}. \tag{9.8}$$

Thus m_0 is a function of $z = e^{-it}$, and as such $m_0 \in L^2(\mathbb{T})$. The orthonormality of $\{\varphi(\,\cdot\, - k)\}$ entails

$$|m_0(z)|^2 + |m_0(-z)|^2 = 2. \tag{9.9}$$

If $W_0 = \mathcal{V}_0^\perp \cap \mathcal{V}_{-1}$, then for a $\xi \in L^2(\mathbb{R})$ it can be shown that $\xi \in W_0$ if and only if ξ has the form

$$\hat{\xi}(2t) = z\bar{m}_0(-z)\,f\left(z^2\right)\hat{\varphi}(t) \tag{9.10}$$

for some function f, where $z = e^{-it}$. Now, *define* ψ as the particular function obtained from φ in this manner with $f = \frac{1}{\sqrt{2}}$, i.e.,

$$\sqrt{2}\hat{\psi}(2t) = z\bar{m}_0(-z)\,\hat{\varphi}(t) = m_1(z)\,\hat{\varphi}(t)\,. \tag{9.11}$$

Then the functions $\psi_{n,k}$ defined by

$$\psi_{n,k}(x) = 2^{-\frac{n}{2}}\psi\left(2^{-n}x - k\right) \tag{9.12}$$

form an orthonormal basis for $L^2(\mathbb{R})$. In fact, it follows from the reasoning in [Dau92] that this does not depend on the specific choice of f above, and any choice

of f such that $|f(z)| = \frac{1}{\sqrt{2}}$ almost everywhere will do. For the specific choice of f we have the explicit expression

$$\psi(x) = \sqrt{2} \sum_k (-1)^k \bar{a}_{1-k} \varphi(2x - k) \tag{9.13}$$

as an orthogonal decomposition. For us it is more important to note that the functions m_0, m_1 satisfy the unitarity condition, i.e., the matrix

$$2^{-\frac{1}{2}} \begin{pmatrix} m_0(z) & m_0(-z) \\ m_1(z) & m_1(-z) \end{pmatrix} \tag{9.14}$$

is unitary for all $z \in \mathbb{T}$. This is indeed the case for any m_1 of the form

$$m_1(z) = z\bar{m}_0(-z) f(z^2) \tag{9.15}$$

where $|f(z)| = 1$ for all z, and, conversely, for m_0 given with (9.9), unitarity of (9.14) implies (9.15).

Conversely, if m_0 satisfies (9.9) and $m_1(z) = z\bar{m}_0(-z)$, iteration of (9.7) and (9.11) give formal product expansions of $\hat{\varphi}$ and $\hat{\psi}$. Moreover, it can be shown [Dau92, Theorem 6.3.6], that if m_0 is a trigonometric polynomial that satisfies $|m_0(t)|^2 + |m_0(t+\pi)|^2 = 2$ and $m_0(0) = \sqrt{2}$, and there exists no nontrivial finite subset $F \subset \mathbb{T}$ with $F^2 \subset F$ such that $m_0|_{-F} = 0$, then φ, ψ defined by

$$\hat{\varphi}(t) = (2\pi)^{-\frac{1}{2}} \prod_{k=1}^{\infty} \left(\frac{m_0(t \cdot 2^{-k})}{\sqrt{2}} \right),$$
$$\hat{\psi}(t) = e^{-\frac{it}{2}} \bar{m}_0\left(\frac{t}{2} + \pi\right) \hat{\varphi}\left(\frac{t}{2}\right)$$

are compactly supported functions in $L^2(\mathbb{R})$ which are the father and mother functions of a wavelet, and in particular

$$\varphi(x) = \sum_k a_k \varphi(2x - k),$$
$$\psi(x) = \sqrt{2} \sum_k (-1)^k \bar{a}_{-k+1} \varphi(2x - k),$$

where a_k are the Fourier coefficients of m_0:

$$m_0(t) = \sum_k a_k e^{-ikt}.$$

Compare this with the different conditions (1.44)–(1.47). The case when such nontrivial subsets F of $\mathbb{T}$, as specified above (and called *cycles*), do exist, is discussed in Remark 12.6 below.

Let us now compute the Hardy-space realization from Theorem 6.2 of the representation of $\mathcal{O}_2$ on $L^2(\mathbb{T})$ defined by (9.14). We must for the moment assume that S_0 is a shift, i.e., by Theorem 3.1 we must assume that there does *not* exist a measurable function $\xi : \mathbb{T} \to \mathbb{T}$ and a $\lambda \in \mathbb{T}$ such that $m(z)\xi(z^2) = \lambda\xi(z)$ for almost all $z \in \mathbb{T}$. We will show later, in Lemma 9.3, that this condition is automatically fulfilled in this situation. The unitary $V : \mathcal{H}_+(\mathcal{K}) \to \mathcal{K}$, where $\mathcal{K} = L^2(\mathbb{T})$, given in general by (6.14), is now defined by

$$V\left(\sum_{k=1}^{\infty} \psi_k z^k\right) = \sum_{k=1}^{\infty} T_k^{\infty} \psi_k \tag{9.16}$$

for $\psi_k \in L^2(\mathbb{T})$ with $\sum_k \|\psi_k\|^2 < \infty$. By (6.6),

$$T_k^\infty = S_0^{k-1} S_1, \quad k = 1, 2, \dots, \tag{9.17}$$

and hence

$$\begin{aligned}(T_k^\infty \xi)(z) &= m_0(z)\, m_0(z^2) \cdots m_0\left(z^{2^{k-2}}\right) m_1\left(z^{2^{k-1}}\right) \xi\left(z^{2^k}\right) \\ &= m_0^{(k-1)}(z)\, m_1\left(z^{2^{k-1}}\right) \xi\left(z^{2^k}\right).\end{aligned} \tag{9.18}$$

(As a general reference to the use of Hardy spaces in operator theory, we give [SzFo70, Chapter V].)

We will connect the Hardy-space description with the wavelet formalism by means of a unitary

$$\mathcal{F}_\varphi : \mathcal{V}_0 \to L^2(\mathbb{T}) = \mathcal{K}, \tag{9.19}$$

an isometric operator

$$M_{\hat\varphi} : \mathcal{K} \to L^2(\hat{\mathbb{R}}), \tag{9.20}$$

and another unitary operator

$$J : L^2(\hat{\mathbb{R}}) \to \mathcal{K} \otimes L^2(\mathbb{T}). \tag{9.21}$$

Let us define these. $\mathcal{F}_\varphi$ is defined by the requirement that it maps $\varphi(\cdot - k)$ into e^{-ikt}, and as $\{\varphi(\cdot - k)\}$ and $\{e^{-ikt}\}$ are orthonormal bases for $\mathcal{V}_0$ and $L^2(\mathbb{T})$ respectively, $\mathcal{F}_\varphi$ is unitary. $M_{\hat\varphi}$ is defined by

$$(M_{\hat\varphi}\xi)(t) = \hat\varphi(t)\, \xi\left(e^{-it}\right). \tag{9.22}$$

We then have

$$\begin{aligned}M_{\hat\varphi}\mathcal{F}_\varphi(\varphi(\cdot - k))(t) &= \hat\varphi(t)\, e^{-ikt} \\ &= \mathcal{F}(\varphi(\cdot - k)),\end{aligned}$$

where $\mathcal{F}$ denotes Fourier transform as defined by (9.6). As $\{\varphi(\cdot - k)\}$ is an orthonormal basis for $\mathcal{V}_0$ this establishes that the diagram

$$\begin{array}{ccc} \mathcal{V}_0 & \xrightarrow{\mathcal{F}_\varphi} & \mathcal{K} = L^2(\mathbb{T}) \\ \big\downarrow & & \big\downarrow M_{\hat\varphi} \\ L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\hat{\mathbb{R}}) \end{array} \tag{9.23}$$

is commutative, and as $\mathcal{F}_\varphi$ and $\mathcal{F}$ are unitaries, it follows that $M_{\hat\varphi}$ is an isometry.

If $\psi_{n,k}$ is the orthonormal basis for $L^2(\mathbb{R})$ given by (9.12), then the Fourier transforms

$$\hat\psi_{n,k}(t) = 2^{\frac{n}{2}} e^{-i2^n kt} \hat\psi(2^n t) \tag{9.24}$$

form an orthonormal basis for $L^2(\hat{\mathbb{R}})$, and we define J by the requirement

$$\left(J\hat\psi_{n,k}\right)\left(e^{-it}, z\right) = e^{-ikt} z^n. \tag{9.25}$$

J maps the orthonormal basis $\hat\psi_{n,k}$ for $L^2(\hat{\mathbb{R}})$ into an orthonormal basis for $\mathcal{K} \otimes L^2(\mathbb{T}) = L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$. If w is a 2π-periodic function we see from (9.25) that

$$J\left(w(\cdot)\, \hat\psi(\cdot)\right)\left(e^{-it}, z\right) = w\left(e^{-it}\right), \tag{9.26}$$

and in particular

$$J\left(\hat{\psi}\right)=1. \tag{9.27}$$

More generally, from (9.25),

$$J\left(2^{\frac{n}{2}}w\left(2^{n}\,\cdot\,\right)\hat{\psi}\left(2^{n}\,\cdot\,\right)\right)=w\left(e^{-it}\right)z^{n}. \tag{9.28}$$

It is also interesting to note that if U is the scaling map given by (9.3) a simple computation shows

$$J\mathcal{F}U\mathcal{F}^{*}J^{*}=M_{z}, \tag{9.29}$$

i.e., U transforms into the operator of multiplication by z. This is because

$$U\psi_{n,k}=\psi_{n+1,k}. \tag{9.30}$$

Let us now connect this to the Hardy-space representation. We have

$$\mathcal{H}_{+}\left(\mathcal{K}\right)=\mathcal{K}\otimes H_{+}^{2}\left(\mathbb{T}\right), \tag{9.31}$$

where $H_{+}^{2}\left(\mathbb{T}\right)$ consists of all vectors in $L^{2}\left(\mathbb{T}\right)$ with a Fourier expansion of the form $\sum_{k=1}^{\infty}a_{k}z^{k}$.

Theorem 9.1. *With the preceding notation and assumptions, the operator $S_0 : L^{2}\left(\mathbb{T}\right)\rightarrow L^{2}\left(\mathbb{T}\right)$ defined by $\left(S_{0}\xi\right)\left(z\right)=m_{0}\left(z\right)\xi\left(z^{2}\right)$ is a shift, and the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{V}_{0} \arrow[r, shift left, "\mathcal{F}_{\varphi}"] \arrow[d, hook] & \mathcal{K}=L^{2}\left(\mathbb{T}\right) \arrow[l, shift left, "\mathcal{F}_{\varphi}^{-1}"] \arrow[r, shift left, "V^{*}"] \arrow[d, "M_{\hat{\varphi}}"] & \mathcal{H}_{+}\left(\mathcal{K}\right)=\mathcal{K}\otimes H_{+}^{2}\left(\mathbb{T}\right) \arrow[l, shift left, "V"] \arrow[d, hook] \\
L^{2}\left(\mathbb{R}\right) \arrow[r, shift left, "\mathcal{F}"] & L^{2}(\hat{\mathbb{R}}) \arrow[l, shift left, "\mathcal{F}^{-1}"] \arrow[r, shift left, "J"] & \mathcal{K}\otimes L^{2}\left(\mathbb{T}\right) \arrow[l, shift left, "J^{-1}"]
\end{tikzcd} \tag{9.32}$$

Proof. We will establish that S_0 is a shift in Lemma 9.3, and hence the map V is well-defined by Theorem 6.2. We have already established commutativity of the left triangle in (9.23), so the right triangle remains, i.e., if $\psi_k\in L^{2}\left(\mathbb{T}\right)=\mathcal{K}$ with $\sum_{k=1}^{\infty}\left\Vert\psi_{k}\right\Vert^{2}<\infty$, we must show

$$JM_{\hat{\varphi}}V\left(\sum_{k=1}^{\infty}{}^{\oplus}\,\psi_{k}\right)\left(t,z\right)=\sum_{k=1}^{\infty}\psi_{k}\left(t\right)z^{k}, \tag{9.33}$$

where $\sum_{k=1}^{\infty}{}^{\oplus}\,\psi_{k}=\sum_{k=1}^{\infty}\psi_{k}z^{k}$. But by (9.15) and (9.18),

$$V\left(\sum_{k=1}^{\infty}{}^{\oplus}\,\psi_{k}\right)\left(t\right)=\sum_{k=1}^{\infty}m_{0}\left(t\right)m_{0}\left(2t\right)\cdots m_{0}\left(2^{k-2}t\right)m_{1}\left(2^{k-1}t\right)\psi_{k}\left(2^{k}t\right), \tag{9.34}$$

so by (9.22) and the iterated versions of (9.7) and (9.11),

$$(9.35)\quad \begin{aligned} (M_{\hat{\varphi}}V)\left(\sum_{k=1}^{\infty}{}^{\oplus}\,\psi_k\right)(t) &= \sum_{k=1}^{\infty}\hat{\varphi}(t)\,m_0(t)\,m_0(2t)\cdots m_0\left(2^{k-2}t\right)m_1\left(2^{k-1}t\right)\psi_k\left(2^k t\right) \\ &= \sum_{k=1}^{\infty}2^{\frac{k}{2}}\hat{\psi}\left(2^k t\right)\psi_k\left(2^k t\right), \end{aligned}$$

where ψ is the mother function. But now apply (9.28) to deduce (9.33). This proves Theorem 9.1. □

Corollary 9.2. *The operator S_0 on $L^2(\mathbb{T})$,*

$$(S_0\xi)(z) = m_0(z)\,\xi\left(z^2\right)$$

is a compression of the scaling operator U on $L^2(\mathbb{R})$,

$$(U\xi)(x) = 2^{-\frac{1}{2}}\xi(x/2),$$

in the sense that

$$(9.36)\quad S_0 = M_{\hat{\varphi}}^{*}\mathcal{F}U\mathcal{F}^{-1}M_{\hat{\varphi}}.$$

Proof. This follows from (6.12) and (9.29). Both operators act as multiplication by z on the respective spaces

$$\mathcal{K}\otimes H_{+}^{2}(\mathbb{T}) \subset \mathcal{K}\otimes L^2(\mathbb{T}).$$ □

Referring to the diagram (9.32) in Theorem 9.1, we will use the term *z-transform* for the map from any vertex into the lower right-hand vertex $\mathcal{K}\otimes L^2(\mathbb{T})$, where z is the variable in $\mathbb{T}$. For example, it follows from (9.25) that the z-transform of $\psi_{n,k}$ is $e^{-ikt}z^n$, and in particular the z-transform of the mother wavelet ψ itself is 1. Let us compute the z-transform $F\left(e^{-it},z\right)$ of the father wavelet φ. Since $\varphi\in\mathcal{V}_0$, it follows by using the three possible paths from (9.32) that F has the form

$$(9.37)\quad \begin{aligned} \sum_{k=1}^{\infty} z^n w_k\left(e^{-it}\right) &= V^{*}\mathcal{F}_{\varphi}(\varphi)\left(e^{-it},z\right) \\ &= JM_{\hat{\varphi}}\mathcal{F}_{\varphi}(\varphi)\left(e^{-it},z\right) \\ &= J\mathcal{F}(\varphi)\left(e^{-it},z\right), \end{aligned}$$

and the first of these identities leads to the following expression for $F\left(e^{-it},z\right)$:

$$(9.38)\quad \begin{aligned} F\left(e^{-it},z\right) &= V^{*}\mathcal{F}_{\varphi}(\varphi)\left(e^{-it},z\right) \\ &= (V^{*}1)\left(e^{-it},z\right) \\ &= \sum_{n=1}^{\infty}\frac{z^n}{2^n}\sum_{\substack{w \\ w^{2^n}=e^{-it}}}\bar{m}_1\left(w^{2^{n-1}}\right)\bar{m}_0^{(n-1)}(w). \end{aligned}$$

Let us compute this expansion for the Haar wavelet

$$(9.39)\quad \varphi(x) = \chi_{[0,1]}(x).$$

Then

$$U(\varphi)(x) = 2^{-\frac{1}{2}}\varphi(x/2) = 2^{-\frac{1}{2}}\varphi(x) + 2^{-\frac{1}{2}}\varphi(x-1), \tag{9.40}$$

so in (9.5) $a_0 = a_1 = 2^{-\frac{1}{2}}$ and all other coefficients are zero. Thus from (9.8) and (9.11),

$$\begin{aligned} m_0(t) &= 2^{\frac{1}{2}} e^{-i\frac{t}{2}} \cos(t/2), \\ m_1(t) &= e^{-it}\bar{m}_0(t+\pi) \\ &= -2^{\frac{1}{2}} e^{-i\frac{t}{2}} \sin(t/2). \end{aligned} \tag{9.41}$$

Hence, from (9.38),

$$F\left(e^{-it}, z\right) = \sum_{n=1}^{\infty}\left(\frac{z}{\sqrt{2}}\right)^n = \frac{z}{\sqrt{2}}\left(1-\frac{z}{\sqrt{2}}\right)^{-1}. \tag{9.42}$$

Note that in this case, from (9.13),

$$\begin{aligned} \psi(x) &= \varphi(2x) - \varphi(2x-1) \\ &= \chi_{\left[0,\frac{1}{2}\right]}(x) - \chi_{\left[\frac{1}{2},1\right]}(x), \end{aligned} \tag{9.43}$$

and the expression (9.42) for the z-transform corresponds to the expansion

$$\begin{aligned} \varphi(x) &= \sum_{n=1}^{\infty} 2^{-\frac{n}{2}}\psi_{n,0}(x) \\ &= \sum_{n=1}^{\infty} 2^{-n}\psi\left(2^{-n}x\right), \end{aligned} \tag{9.44}$$

which can be verified by hand.

Finally, let us compute (9.42) by combining (9.43) with the obvious relation

$$\varphi(x) = \varphi(2x) + \varphi(2x-1).$$

Adding these, we see

$$2\varphi(2x) = \varphi(x) + \psi(x),$$

i.e.,

$$2^{\frac{1}{2}}\varphi = U\varphi + U\psi.$$

Now take the z-transform and use (9.29) to obtain

$$2^{\frac{1}{2}}F = zF + z,$$

which gives (9.42). In general it seems more difficult to obtain simple expressions for F using (9.5) and (9.13), $U\varphi = \sum_k a_k\varphi(\,\cdot\,-k)$, $U\psi = \sum_k (-1)^k \bar{a}_{1-k}\varphi(\,\cdot\,-k)$, since the z-transform does not have any particularly simple property with respect to translation by 1 in $L^2(\mathbb{R})$, and a derivation of F along these lines leads back to (9.38).

Lemma 9.3. *The function m_0 defined from the father function φ by* (9.5) *and* (9.8) *has the property that $|m_0(z)| \neq 1$ for a set z of positive measure. In particular the operator $S_0 : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by $(S_0\xi)(z) = m_0(z)\,\xi\left(z^2\right)$ is a shift.*

Proof. The last statement follows from the former by Theorem 3.1, and hence we only need to show that we cannot have $|m_0(z)| = 1$ almost everywhere. If *ad absurdum* this is the case, it follows from (9.7) that

$$\sqrt{2}\,|\hat{\varphi}(2t)| = |\hat{\varphi}(t)|$$

for almost all $t \in \mathbb{R}$. This means that $|\hat{\varphi}|$ is an eigenvector with eigenvalue 1 of the unitary operator $\hat{U}$ on $L^2(\mathbb{R})$ defined by

$$\left(\hat{U}\xi\right)(t) = \sqrt{2}\xi(2t).$$

But this unitary is a multiple of the two-sided shift by the following reasoning: we have a decomposition $L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$ into $\hat{U}$- and $\hat{U}^*$-invariant subspaces, and it suffices to consider $L^2(\mathbb{R}_+)$. One checks that $V : L^2(\mathbb{R}_+, dt) \to L^2(\mathbb{R}, ds)$ defined by $(V\eta)(s) = \eta(e^s)\, e^{\frac{s}{2}}$ for $\eta \in L^2(\mathbb{R}_+, dt)$ is unitary and

$$\left(V\hat{U}V^*\xi\right)(s) = \xi(s + \ln 2)$$

for $\xi \in L^2(\mathbb{R}, ds)$. If one furthermore defines a map $W : L^2(\mathbb{R}) \to L^2(\mathbb{T} \times [0, 2\pi\rangle)$ by

$$(W\xi)(z)(s) = \sum_{k=-\infty}^{\infty} z^{-k}\xi(s + k\ln 2),$$

one checks that

$$WV\hat{U} = M_z WV,$$

i.e., $\hat{U}$ is unitarily equivalent with multiplication by z on $L^2(\mathbb{T} \times [0, \ln 2\rangle) \otimes \mathbb{C}^2$, where z is the $\mathbb{T}$ variable. But this operator has absolutely continuous spectrum, so we cannot have 1 as a discrete eigenvalue. Thus $|m_0(z)| \neq 1$ on a set of z of positive measure. □

10. Wavelets of scale N

The construction in Section 9 can be generalized in various directions. One generalization which is well known in wavelet theory is to replace the strict orthogonality requirement (9.1) on the translates of φ by a weaker requirement like, say,

$$\left\| \sum_{n\in\mathbb{Z}} \xi_n \varphi(\,\cdot - n) \right\|_{L^2(\mathbb{R})}^2 \leq c\,\|\xi\|_{\ell^2}^2$$

for all $\xi = (\xi_n)_{n\in\mathbb{Z}}$ in $\ell^2 = \ell^2(\mathbb{Z})$. This will be considered in Section 12, and has interest when going from $\mathcal{O}_2$-representations back to wavelets. But before that we will consider another generalization which is interesting for us but seems to have been merely postulated in wavelet theory without proper proofs [GrMa92], [Mey87], [MRF96]: the replacement of scale 2 by scale N, with $N \in \{3, 4, 5, \dots\}$. In this case we still start with a father function φ with the properties (9.1),(9.3), and (9.4) replaced with their natural generalizations

(10.1) $\{\varphi(\,\cdot - k) \mid k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$,

$$\mathcal{V}_0 = \overline{\operatorname{span}} \left\{ \varphi \left(\,\cdot\, - k \right) \right\},$$

$$\left(U_N \xi \right) (x) = N^{-\frac{1}{2}} \xi \left(x/N \right),$$

$$U_N \varphi \in \mathcal{V}_0, \tag{10.2}$$

$$\bigwedge_n U_N^n \mathcal{V}_0 = \{0\}, \tag{10.3a}$$

$$\bigvee_n U_N^n \mathcal{V}_0 = L^2 \left(\mathbb{R} \right). \tag{10.3b}$$

Again, define $(a_n) \in \ell_2$ by

$$U_N \varphi = \sum_k a_k \varphi \left(\,\cdot\, - k \right), \tag{10.4}$$

i.e.,

$$\sqrt{N} \hat{\varphi} \left(Nt \right) = m_0 \left(t \right) \hat{\varphi} \left(t \right), \tag{10.5}$$

where

$$m_0 \left(t \right) = \sum_k a_k e^{-ikt}. \tag{10.6}$$

Thus m_0 may be viewed as a function on $\mathbb{T}$. As in [Dau92, (5.1.20)], the orthonormality of $\varphi \left(\,\cdot\, - k \right)$ is now, by Fourier transform, equivalent to the condition

$$\operatorname{PER} \left(\left| \hat{\varphi} \right|^2 \right) (t) := \sum_k \left| \hat{\varphi} \left(t + 2\pi k \right) \right|^2 = \left(2\pi \right)^{-1} \tag{10.7}$$

for almost all t. Also, using (10.5), one has

$$\begin{aligned} \operatorname{PER} & \left(\left| \hat{\varphi} \right|^2 \right) (t) \\ &= \frac{1}{N} \sum_k \left| m_0 \left(\frac{t + 2\pi k}{N} \right) \hat{\varphi} \left(\frac{t + 2\pi k}{N} \right) \right|^2 \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n \in \mathbb{Z}} \left| m_0 \left(\frac{t + 2\pi m}{N} + 2\pi n \right) \right|^2 \left| \hat{\varphi} \left(\frac{t + 2\pi m}{N} + 2\pi n \right) \right|^2 \\ &= \frac{1}{N} \sum_{\substack{w \\ w^N = e^{-it}}} \left| m_0 \left(w \right) \right|^2 \operatorname{PER} \left(\left| \hat{\varphi} \right|^2 \right) (w), \end{aligned} \tag{10.8}$$

and combining this with (10.7) we see that orthonormality of $\left\{ \varphi \left(\,\cdot\, - k \right) \right\}$ implies

$$\sum_{k \in \mathbb{Z}_N} \left| m_0 \left(t + 2\pi k/N \right) \right|^2 = N. \tag{10.9}$$

By the example on pages 177–178 of [Dau92], the relation (10.9) does not conversely imply that $\left\{ \varphi \left(\,\cdot\, - k \right) \right\}$ is orthonormal. If we define an operator $R : L^\infty \left(\mathbb{T} \right) \to L^\infty \left(\mathbb{T} \right)$ by

$$\left(R \xi \right) (z) = \frac{1}{N} \sum_{\substack{w \\ w^N = z}} \left| m_0 \left(w \right) \right|^2 \xi \left(w \right), \tag{10.10}$$

then we see that (10.9) together with the assumption that the eigensubspace of R corresponding to eigenvalue 1 is one-dimensional, implies that (10.7) holds, i.e., $\left\{ \varphi \left(\,\cdot\, - k \right) \right\}$ is orthonormal. Conversely, one can use the ergodicity of $z \mapsto z^N$ to

show that (10.7) implies the eigensubspace of R corresponding to eigenvalue 1 is one-dimensional; see Section 12.

More generally, if $\xi \in \mathcal{V}_{-1} = U_N^{-1}\mathcal{V}_0$ then $U_N\xi$ has a decomposition

$$U_N\xi = \sum_k \xi_k \varphi\left(\,\cdot\, - k\right), \tag{10.11}$$

and defining

$$m_\xi\left(t\right) = \sum_k \xi_k e^{-ikt} \tag{10.12}$$

we have

$$\sqrt{N}\hat{\xi}\left(Nt\right) = m_\xi\left(t\right)\hat{\varphi}\left(t\right). \tag{10.13}$$

Define the operator T on $L^2\left(\mathbb{R}\right)$ by

$$\left(T\xi\right)\left(x\right) = \xi\left(x-1\right). \tag{10.14}$$

If $\xi, \eta \in \mathcal{V}_{-1}$ one then uses (10.13) and (10.7) to compute, as in [Dau92, (5.1.21)–(5.1.24)], that the vectors ξ and $T^k\eta$ are orthogonal for all $k \in \mathbb{Z}$ if and only if

$$\sum_{k\in\mathbb{Z}_N} m_\xi\left(t + 2\pi k/N\right)\bar{m}_\eta\left(t + 2\pi k/N\right) = 0 \tag{10.15}$$

for almost all $t \in \mathbb{R}$, and also $\{\xi\left(\,\cdot\, - k\right)\}$ is an orthonormal set if and only if

$$\sum_{k\in\mathbb{Z}_N} \left|m_\xi\left(t + 2\pi k/N\right)\right|^2 = N. \tag{10.16}$$

For the latter statement, one uses (10.13) and the same computation as in (10.8) to show

$$\begin{aligned} \operatorname{PER}\left(|\hat{\xi}|^2\right)(z) &= R\left(\operatorname{PER}\left(|\hat{\varphi}|^2\right)\right) \\ &= R\left(\left(2\pi\right)^{-1}1\right) \\ &= \frac{1}{2\pi N}\sum_{\substack{w \\ w^N = z}} \left|m_\xi\left(w\right)\right|^2, \end{aligned}$$

and hence $\operatorname{PER}\left(|\hat{\xi}|^2\right) = \left(2\pi\right)^{-1}$ almost everywhere if and only if (10.16) holds. Here R is defined by m_ξ rather than m_0 as in (10.10). Thus, if $\xi \in \mathcal{V}_{-1}$, the vectors $T^k\xi$ are mutually orthogonal if and only if the function $\vec{m}_\xi : \mathbb{T} \to \mathbb{C}^N$ defined by

$$\vec{m}_\xi\left(z\right) = \left(m_\xi\left(z\right), m_\xi\left(\rho_N z\right), \dots, m_\xi\left(\rho_N^{N-1}z\right)\right), \tag{10.17}$$

where $\rho_N = e^{\frac{2\pi i}{N}}$, takes values in the sphere of radius $N^{\frac{1}{2}}$ for almost all z, and $T^k\xi$ and $T^l\eta$ are mutually orthogonal if and only if

$$\langle \vec{m}_\xi(z) \,|\, \vec{m}_\eta(z)\rangle = 0 \tag{10.18}$$

for almost all z.

Now given $m_\xi : \mathbb{T} \to \mathbb{C}$ with the property (10.16), the corresponding $\xi \in L^2\left(\mathbb{R}\right)$ can be defined from the relations (10.11)–(10.13). In this way one may construct a set of functions $\psi_1, \dots, \psi_{N-1}$ in $L^2\left(\mathbb{R}\right)$ such that if $m_i\left(z\right) = m_{\psi_i}\left(z\right)$ for $i = 1, \dots, N-1$ and $m_0\left(z\right)$ is given by (10.9), then the vectors

$$N^{-\frac{1}{2}}\vec{m}_0\left(z\right), N^{-\frac{1}{2}}\vec{m}_1\left(z\right), \dots, N^{-\frac{1}{2}}\vec{m}_{N-1}\left(z\right)$$

form an orthonormal basis of $\mathbb{C}^N$ for each $z \in \mathbb{T}$. This can for example be done by choosing a fixed measurable map F from the unit sphere S^{2N-1} in $\mathbb{C}^N$ into N-dimensional orthogonal frames in $\mathbb{C}^N$, i.e.,

$$F(\vec{x}) = \left(\vec{F}_0(\vec{x}), \dots, \vec{F}_{N-1}(\vec{x})\right), \tag{10.19}$$

where the vectors $\vec{F}_0(\vec{x}), \dots, \vec{F}_{N-1}(\vec{x})$ form an orthonormal basis and we assume $\vec{F}_0(\vec{x}) = \vec{x}$. It is no problem finding such measurable maps, but they can be chosen continuous if and only if $N = 2, 4, 8$; see Remark 10.2 below. Having chosen F, one cannot now just set

$$\vec{m}_i(z) = N^{\frac{1}{2}} \vec{F}_i\left(N^{-\frac{1}{2}} \vec{m}_0(z)\right), \tag{10.20}$$

as this may break the particular symmetry enjoyed by the vector functions of the form (10.17), that is,

$$\vec{m}(\rho z) = V \vec{m}(z), \tag{10.21}$$

where V is the permutation matrix

$$V = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{10.22}$$

So one defines $\vec{m}_i(z)$ by (10.20) just when $0 \le \operatorname{Arg} z < 2\pi/N$, and then extend the definition to all z by requiring (10.21). Since V is unitary it follows that the ensuing functions $N^{-\frac{1}{2}} \vec{m}_0(z), \dots, N^{-\frac{1}{2}} \vec{m}_{N-1}(z)$ form an orthonormal basis for each $z \in \mathbb{T}$.

Thus we have established most of the following probably known extension of [Dau92, Theorem 5.1.1], but we have not found the result explicitly in the literature, other than as a postulate [GrMa92], [Mey87], [MRF96].

Theorem 10.1. *Let $\varphi \in L^2(\mathbb{R})$ be a function satisfying* (10.1), (10.2), *and* (10.3). *Then there exist $N-1$ functions $\psi_1, \dots, \psi_{N-1}$ such that $\left\{T^k \psi_m\right\}$, $k \in \mathbb{Z}$, $m = 1, \dots, N-1$ forms an orthogonal basis for $\mathcal{V}_0^\perp \cap \mathcal{V}_{-1}$, and thus $\left\{U_N^n T^k \psi_m\right\}$, $n, k \in \mathbb{Z}$, $m = 1, \dots, N-1$ forms an orthonormal basis for $L^2(\mathbb{R})$. Furthermore, the sequences $\psi_1, \dots, \psi_{N-1}$ of such mother functions are in one-to-one correspondence with the sequences $m_1, \dots, m_{N-1}$ of functions in $L^2(\mathbb{T})$ with the property that $N^{-\frac{1}{2}} \vec{m}_0(z), N^{-\frac{1}{2}} \vec{m}_1(z), \dots, N^{-\frac{1}{2}} \vec{m}_{N-1}(z)$ is an orthonormal set for almost all $z \in \mathbb{T}$. The correspondence is given by*

$$\sqrt{N} \hat{\psi}_k(Nt) = m_k\left(e^{-it}\right) \hat{\varphi}(t) \tag{10.23}$$

for $k = 1, \dots, N-1$. Furthermore, if $\psi_1, \dots, \psi_M$ is any sequence in $\mathcal{V}_0^\perp \cap \mathcal{V}_{-1}$ such that $\left\{T^k \psi_m\right\}$ forms an orthonormal set, then $M \le N-1$, and $\left\{T^k \psi_m\right\}$ then is an orthonormal basis if and only if $M = N-1$.

Proof. The only thing remaining to prove is that $\left\{T^k \psi_m\right\}$, $k \in \mathbb{Z}$, $m = 1, \dots, N-1$ really forms a basis for $\mathcal{V}_0^\perp \cap \mathcal{V}_{-1}$ when ψ_m is constructed as before. But any $\xi \in \mathcal{V}_{-1}$ has the form

$$\sqrt{N} \hat{\xi}(Nt) = m(t) \hat{\varphi}(t),$$

where $m \in L^2(\mathbb{T})$, by the reasoning leading to (10.13), and ξ being orthogonal to $\mathcal{V}_0$ means

$$\sum_{k=1}^{N-1} m\left(\rho^k z\right) \bar{m}_0\left(\rho^k z\right) = 0$$

by the reasoning leading to (10.15). But this means that $\vec{m}(z)$ is orthogonal to $\vec{m}_0(z)$ for almost all $z \in \mathbb{T}$, and it follows that $\vec{m}(z)$ is a linear combination of $\vec{m}_1(z), \dots, \vec{m}_{N-1}(z)$ for almost all z:

$$\vec{m}(z) = \sum_{k=1}^{N-1} \mu_k(z) \vec{m}_k(z).$$

The symmetries

$$\vec{m}(\rho z) = V\vec{m}(z),$$
$$\vec{m}_k(\rho z) = V\vec{m}_k(z)$$

imply that

$$\mu_k(\rho z) = \mu_k(z),$$

and hence

$$\mu_k(z) = \lambda_k\left(z^N\right)$$

for a suitable function λ_k on $\mathbb{T}$. Thus

$$\begin{aligned}\sqrt{N}\hat{\xi}(Nt) &= \sum_{k=1}^{N-1} \lambda_k(Nt) m_k(t) \hat{\varphi}(t) \\ &= \sum_{k=1}^{N} \lambda_k(Nt) \sqrt{N}\hat{\psi}_k(Nt),\end{aligned}$$

i.e.,

$$\hat{\xi}(t) = \sum_{k=1}^{N} \lambda_k(t) \hat{\psi}_k(t).$$

But putting

$$\lambda_k(t) = \sum_m a_m^{(k)} e^{-ikt},$$

this means

$$\xi = \sum_{k,m} a_m^{(k)} \psi_k(\cdot - m),$$

so $\{T^m \psi_k\}$ is a basis. □

Remark 10.2. In Daubechies's approach to wavelets, the selection function F in (10.19) has a particularly simple form like

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10.24}$$

When $N > 2$ one cannot always choose the selection function this simple, and by a celebrated theorem of Adams, it cannot even be chosen continuous except in the

cases $N=2,4,8$. If S^{n-1} is the unit sphere in $\mathbb{R}^n$, Adams's theorem [Ada62] says that the highest number of pointwise linearly independent vector fields that may be defined on S^{n-1} is $\rho(n)-1$, where the function $\rho(n)$ is defined as follows: let b be the multiplicity of 2 in the prime decomposition of n; write $b=c+4d$ where $c\in\{0,1,2,3\}$ and $d\in\{0,1,2,\dots\}$; and put $\rho(n)=2^c+8d$. One checks that $\rho(2N)-1\geq N$ if and only if $N=2,4,8$ (where $\rho(2N)-1=3,7,8$). In the cases $N=2,4,8$ where the maps F can be chosen continuous, they may actually be chosen very simple: if $N=2$, let F_1 be multiplication by i on $\mathbb{R}^2=\mathbb{C}$; if $N=4$, let F_1, F_2, F_3 be multiplication by i, j, k respectively on the quaternions; and when $N=8$ use the same method with the Cayley numbers, and then view the resulting (real) orthogonal $N\times N$ matrices as unitary matrices.

At this stage it is no surprise that the appropriate analogue of Theorem 9.1 is also true in this more general setting. If $\mathcal{K}=L^2(\mathbb{T})$, define the unitary

$$V:\mathcal{H}_+\left(\bigoplus_{j=1}^{N-1}\mathcal{K}\right)\to\mathcal{K} \tag{10.25}$$

as in (6.14). Put $\bigoplus_{j=1}^{N-1}\mathcal{K}=\mathbb{C}^{N-1}\otimes\mathcal{K}$ in such a way that the j'th component of $\bigoplus_{j=1}^{N-1}\mathcal{K}$ identifies with $e_j\otimes\mathcal{K}$, where $\{e_1,\dots,e_{N-1}\}$ is the standard basis for $\mathbb{C}^{N-1}$. In this case we define the unitary map

$$J:L^2(\hat{\mathbb{R}})\to\mathbb{C}^{N-1}\otimes\mathcal{K}\otimes L^2(\mathbb{T}) \tag{10.26}$$

by the requirement

$$J\left(U_N^nT^k\psi_m\right)\left(e^{-it},z\right)=e_m\otimes e^{-ikt}\otimes z^n. \tag{10.27}$$

The following result is now proved exactly as Theorem 9.1:

Corollary 10.3. *With the preceding notation and assumptions, the operator $S_0: L^2(\mathbb{T})\to L^2(\mathbb{T})$ defined by $(S_0\xi)(z)=m_0(z)\,\xi\left(z^N\right)$ is a shift, and the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{V}_0 & \underset{\mathcal{F}_\varphi^{-1}}{\overset{\mathcal{F}_\varphi}{\rightleftarrows}} & \mathcal{K}=L^2(\mathbb{T}) & \underset{V}{\overset{V^*}{\rightleftarrows}} & \mathbb{C}^{N-1}\otimes\mathcal{K}\otimes H_+^2(\mathbb{T}) \\ \downarrow & & \downarrow{\scriptstyle M_{\hat{\varphi}}} & & \downarrow \\ L^2(\mathbb{R}) & \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} & L^2(\hat{\mathbb{R}}) & \underset{J^*}{\overset{J}{\rightleftarrows}} & \mathbb{C}^{N-1}\otimes\mathcal{K}\otimes L^2(\mathbb{T}) \end{array} \tag{10.28}$$

In particular, the operator S_0 is a compression of the scaling operator $(U_N\xi)(x)=N^{-\frac{1}{2}}\xi(x/N)$ in the sense

$$S_0=M_{\hat{\varphi}}^*\mathcal{F}U_N\mathcal{F}^{-1}M_{\hat{\varphi}}, \tag{10.29}$$

both operators being represented by multiplication by z by the z-transform.

We end this section by proving the formula (1.35) in the Introduction.

Corollary 10.4. *Adopt the preceding notation and assumptions. For any $\xi\in L^2(\mathbb{R})$, let*

$$\xi=\sum_{i=1}^{N-1}\sum_{j,k\in\mathbb{Z}}a_{jk}^{(i)}(\xi)\,U_N^jT^k\psi_i$$

be the orthonormal decomposition of ξ *relative to the wavelet basis in Theorem* 10.1, *and if* $\xi \in \mathcal{V}_0$ *(i.e.,* $a_{jk}^{(i)}(\xi) = 0$ *for* $j \le 0$*), let* $f \in L^2(\mathbb{T}) = L^2(\mathbb{R}/2\pi\mathbb{Z})$ *be the unique function such that*

$$\hat{\xi}(t) = f(t)\,\hat{\varphi}(t).$$

Then

$$a_{jk}^{(i)}(\xi) = \left(S_i^* S_0^{*\,j-1} f\right)^{\sim}(k)$$

for $i = 1, \dots, N-1$, $j = 1, 2, \dots$, $k \in \mathbb{Z}$, *where* $(\,\cdot\,)^{\sim}$ *refers to the Fourier transform* (1.36) *on* $L^2(\mathbb{T})$.

Proof. We have

$$\begin{aligned} a_{jk}^{(i)}(\xi) &= N^{-\frac{j}{2}} \int_{\mathbb{R}} \bar{\psi}_i\left(N^{-j}x - k\right)\xi(x)\,dx \\ &= N^{\frac{j}{2}} \int_{\hat{\mathbb{R}}} e_i^{+ikN^j t}\,\bar{\hat{\psi}}_i\left(N^j t\right) f(t)\,\hat{\varphi}(t)\,dt \\ &= N^{\frac{j}{2}} \int_{\mathbb{T}} e^{+ikN^j t} f(t)\,\mathrm{PER}\left(\bar{\hat{\psi}}_i\left(N^j\,\cdot\,\right)\hat{\varphi}(\,\cdot\,)\right)(t)\,dt. \end{aligned}$$

Using (10.23) and then (10.5) iteratively, we have further

$$\begin{aligned} a_{jk}^{(i)}(\xi) &= \int_{\mathbb{T}} e^{+ikN^j t} f(t)\,\bar{m}_0\left(N^{j-1}t\right)\bar{m}_i\left(N^{j-2}t\right)\cdots\bar{m}_i(Nt)\,\bar{m}_i(t)\,\mathrm{PER}\left(\bar{\hat{\varphi}}\hat{\varphi}\right)(t)\,dt \\ &= (2\pi)^{-1}\int_{\mathbb{T}} e^{+ikN^j t} f(t)\,\bar{m}_0\left(N^{j-1}t\right)\cdots\bar{m}_i(t)\,dt, \end{aligned}$$

where the last step used the orthonormality of $\{\varphi(\,\cdot - k)\}$ in the form (10.7). Introducing $e_k(t) = e^{-ikt}$, or in complex form $e_k(z) = z^k$, we furthermore compute

$$\begin{aligned} a_{jk}^{(i)}(\xi) &= \frac{1}{2\pi}\int_{\mathbb{T}} \overline{S_0^{j-1} S_i(e_k)(t)}\,f(t)\,dt \\ &= \left\langle S_0^{j-1} S_i(e_k) \,\middle|\, f \right\rangle_{L^2(\mathbb{T})} \\ &= \left\langle e_k \,\middle|\, S_i^* S_0^{*\,j-1} f \right\rangle_{L^2(\mathbb{T})} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}\left(S_i^* S_0^{*\,j-1} f\right)(t)\,dt, \end{aligned}$$

which is the desired conclusion. ∎

Generalizations of these results to non-orthogonal translates of the father wavelet will be given in Section 12.

11. Scattering theory for scale 2 versus scale N

Let $\Phi \in L^2(\mathbb{R})$ be a scale-N father function as introduced in (10.1)–(10.3). By iteration of (10.5) we have

$$\hat{\Phi}(t) = \prod_{k=1}^{n}\left(N^{-\frac{1}{2}} M_0\left(tN^{-k}\right)\right)\hat{\Phi}(t/N^n), \tag{11.1}$$

where we now denote the function m_0 by M_0. It is known (by Remark 3 following Proposition 5.3.2 in [Dau92]) that $\hat{\Phi}(0) \neq 0$, and thus $M_0(0) = N^{\frac{1}{2}}$ by (10.5).

Thus, if we assume that $\hat{\Phi}$ is continuous at 0, it follows from (11.1) and the normalization (10.7) that

$$\hat{\Phi}(t) = (2\pi)^{-\frac{1}{2}} \prod_{k=1}^{\infty} \left(N^{-\frac{1}{2}} M_0 \left(t N^{-k} \right) \right), \tag{11.2}$$

at least up to a phase factor, but we choose the latter to be 1. Correspondingly, if φ is a scale-2 father function, we have

$$\hat{\varphi}(t) = (2\pi)^{-\frac{1}{2}} \prod_{k=1}^{\infty} \left(2^{-\frac{1}{2}} m_0 \left(t 2^{-k} \right) \right) \tag{11.3}$$

under slight regularity assumptions, where m_0 is now defined by (9.8). Throughout this section we will assume that Φ and φ are sufficiently regular that (11.2) and (11.3) are valid. (See also the discussion at the end of Section 1.) Denote the associated isometries defined by (1.16) by T_0 and S_0, respectively, i.e.,

$$(T_0 \xi)(z) = M_0(z) \xi \left(z^N \right), \tag{11.4}$$

$$(S_0 \xi)(z) = m_0(z) \xi \left(z^2 \right). \tag{11.5}$$

Proposition 11.1. *Adopt the notation and assumptions above. The following three conditions are equivalent.*

(11.6) $\varphi = \Phi$.

(11.7) $m_0(Nt) M_0(t) = m_0(t) M_0(2t)$ *for almost all* $t \in \mathbb{R}$.

(11.8) $S_0 T_0 = T_0 S_0$.

Proof. (11.7)$\Leftrightarrow$(11.8): If $\xi \in L^2(\mathbb{T})$ then

$$\begin{aligned} (S_0 T_0 \xi)(z) &= m_0(z) (T_0 \xi) \left(z^2 \right) \\ &= m_0(z) M_0 \left(z^2 \right) \xi \left(z^{2N} \right) \end{aligned}$$

and

$$\begin{aligned} (T_0 S_0 \xi)(z) &= M_0(z) (S_0 \xi) \left(z^N \right) \\ &= M_0(z) m_0 \left(z^N \right) \xi \left(z^{2N} \right), \end{aligned}$$

so (11.7)$\Leftrightarrow$(11.8) is immediate.

(11.6)$\Rightarrow$(11.7): If $\Phi = \varphi$, it follows from (11.1) with $n = 1$ and $N = 2, N$ that

$$\begin{aligned} \hat{\varphi}(t) &= 2^{-\frac{1}{2}} m_0(t/2) \hat{\varphi}(t/2) \\ &= 2^{-\frac{1}{2}} m_0(t/2) N^{-\frac{1}{2}} M_0(t/2N) \hat{\varphi}(t/2N) \end{aligned}$$

and

$$\begin{aligned} \hat{\varphi}(t) &= N^{-\frac{1}{2}} M_0(t/N) \hat{\varphi}(t/N) \\ &= N^{-\frac{1}{2}} M_0(t/N) 2^{-\frac{1}{2}} m_0(t/2N) \hat{\varphi}(t/2N), \end{aligned}$$

and (11.7) is immediate.

(11.7)⇒(11.6): Assuming (11.7) we have

$$\begin{aligned}
M_0(t)\,\hat{\varphi}(t) &= (2\pi)^{-\frac{1}{2}}\,M_0(t)\,2^{-\frac{1}{2}}m_0(t/2)\prod_{k=2}^{\infty}\left(2^{-\frac{1}{2}}m_0\left(t2^{-k}\right)\right)\\
&= (2\pi)^{-\frac{1}{2}}\,2^{-\frac{1}{2}}m_0(Nt/2)\,M_0(t/2)\prod_{k=2}^{\infty}\left(2^{-\frac{1}{2}}m_0\left(t2^{-k}\right)\right)\\
&= (2\pi)^{-\frac{1}{2}}\,2^{-\frac{2}{2}}m_0(Nt/2)\,m_0\left(Nt/2^2\right)M_0\left(t/2^2\right)\prod_{k=3}^{\infty}\left(2^{-\frac{1}{2}}m_0\left(t2^{-k}\right)\right)\\
&= \dots\\
&= (2\pi)^{-\frac{1}{2}}\prod_{k=1}^{n}\left(2^{-\frac{1}{2}}m_0\left(Nt2^{-k}\right)\right)M_0\left(t2^{-n}\right)\prod_{k=n+1}^{\infty}\left(2^{-t}m_0\left(t2^{-k}\right)\right)\\
&\underset{n\to\infty}{\longrightarrow} \hat{\varphi}(Nt)\,M_0(0) = N^{\frac{1}{2}}\hat{\varphi}(Nt)
\end{aligned}$$

Thus

$$N^{\frac{1}{2}}\hat{\varphi}(Nt) = M_0(t)\,\hat{\varphi}(t)\,.$$

But from this one deduces in the same way as (11.2) that

$$\hat{\varphi}(t) = (2\pi)^{-\frac{1}{2}}\prod_{k=1}^{\infty}\left(N^{-\frac{1}{2}}M_0\left(tN^{-k}\right)\right),$$

and hence

$$\hat{\varphi}(t) = \hat{\Phi}(t)\,.$$ □

Proposition 11.1 gives a characterization of the scale-2 father functions φ which are also of scale N. Next, assume that φ is only a scale-2 father function satisfying (9.1)–(9.4), define m_0 by (9.8), and next define m_1 by (9.11), i.e.,

$$m_1(z) = z\bar{m}_0(-z)\,. \tag{11.9}$$

Let ψ be the corresponding mother function defined by (9.11) or (9.13). Then we have the orthogonal decomposition

$$N^{-\frac{1}{2}}\varphi(x/N) = \sum_k A_k\varphi(x-k) + \sum_k B_k\psi(x-k) + \xi_-(x)\,, \tag{11.10}$$

where $\xi_- \in \mathcal{V}_{-1}^{\perp}$ and $\sum_k\left(|A_k|^2+|B_k|^2\right)+\|\xi_-\|_2^2 = 1$. Define

$$A(t) = \sum_k A_k e^{-ikt},\quad B(t) = \sum_k B_k e^{-ikt}. \tag{11.11}$$

Proposition 11.2. *If φ is a scale-2 father function we have, with the notation introduced above,*

$$A(2t)\,m_0(t) + B(2t)\,m_1(t) = A(t)\,m_0(Nt)\,, \tag{11.12}$$

or, in terms of the representation S_0, S_1 of $\mathcal{O}_2$ defined by φ,

$$S_0(A)(z) + S_1(B)(z) = m_0\left(z^N\right)A(z)\,. \tag{11.13}$$

In particular, the functions in the left sum are orthogonal, so

$$\|A\|_{L^2(\mathbb{T})}^2+\|B\|_{L^2(\mathbb{T})}^2=\int_{\mathbb{T}}\left|m_0\left(z^N\right)A\left(z\right)\right|^2\frac{|dz|}{2\pi}$$
$$=N^{-1}\int_{\mathbb{T}}\left|m_0\left(z\right)\right|^2\sum_{\substack{w\\w^N=z}}\left|A\left(w\right)\right|^2\frac{|dz|}{2\pi}. \tag{11.14}$$

Proof. By Fourier transform of (11.10) we have

$$N^{\frac{1}{2}}\hat{\varphi}\left(Nt\right)=A\left(t\right)\hat{\varphi}\left(t\right)+B\left(t\right)\hat{\psi}\left(t\right)+\hat{\xi}_-\left(t\right). \tag{11.15}$$

Thus, by (9.7) and (11.15),

$$\begin{aligned}(2N)^{\frac{1}{2}}\hat{\varphi}\left(2Nt\right)&=N^{\frac{1}{2}}m_0\left(Nt\right)\hat{\varphi}\left(Nt\right)\\&=m_0\left(Nt\right)\Big(A\left(t\right)\hat{\varphi}\left(t\right)+B\left(t\right)\hat{\psi}\left(t\right)+\hat{\xi}_-\left(t\right)\Big).\end{aligned} \tag{11.16}$$

On the other hand, by (11.15), (9.7), and (9.11),

$$\begin{aligned}(2N)^{\frac{1}{2}}\hat{\varphi}\left(2Nt\right)&=2^{\frac{1}{2}}\Big(A\left(2t\right)\hat{\varphi}\left(2t\right)+B\left(2t\right)\hat{\psi}\left(2t\right)+\hat{\xi}_-\left(2t\right)\Big)\\&=A\left(2t\right)m_0\left(t\right)\hat{\varphi}\left(t\right)+B\left(2t\right)m_1\left(t\right)\hat{\varphi}\left(t\right)+2^{\frac{1}{2}}\hat{\xi}_-\left(2t\right).\end{aligned} \tag{11.17}$$

Now, applying the orthogonal projection onto $\hat{\mathcal{V}}_0$ on (11.16) and (11.17) and equating the two expressions, we obtain

$$m_0\left(Nt\right)A\left(t\right)\hat{\varphi}\left(t\right)=\left(A\left(2t\right)m_0\left(t\right)+B\left(2t\right)m_1\left(t\right)\right)\hat{\varphi}\left(t\right)$$

for almost all $t\in\mathbb{R}$. But multiplying both sides by $2\pi\bar{\hat{\varphi}}\left(t\right)$ and adding over all $t:=t+2\pi k$, $k\in\mathbb{Z}$, using (10.7), we obtain (11.12). The formula (11.13) is just a transcription of (11.12) (using (1.16) with $N=2$), and since S_0 and S_1 are isometries with orthogonal ranges, (11.14) follows. ☐

Scholium 11.3. Note in particular that $\mathcal{V}_0$ is invariant under scaling by N if and only if $B=0$ and $\xi_-=0$, and then $A\left(t\right)=M_0\left(t\right)$, and (11.13) reduces to the relation (11.7). Thus B is a measure of the non-N-scale invariance of $\mathcal{V}_0$.

By Theorem 9.1, to apply the projection onto $\hat{\mathcal{V}}_0$ is equivalent to projecting onto the vectors of the form $\sum_{n=1}^{\infty}\xi_nz^n$ in the z-transformed Hilbert space. In this space ξ_- has the form

$$\hat{\xi}_-\sim z^{-1}C_1+z^{-2}C_2+\cdots \tag{11.18}$$

while

$$m_0\left(Nt\right)B\left(t\right)\hat{\psi}\left(t\right)\sim m_0\left(Nt\right)B\left(t\right) \tag{11.19}$$

by (9.28). But (9.28) implies that

$$\begin{aligned}J\left(w\left(\,\cdot\,\right)\hat{\psi}_{n,k}\left(\,\cdot\,\right)\right)&=J\left(w\left(2^{-n}2^n\,\cdot\,\right)e^{-i2^nk\,\cdot}\,2^{\frac{n}{2}}\hat{\psi}\left(2^n\,\cdot\,\right)\right)\\&=w\left(2^{-n}t\right)e^{-ikt}z^n\end{aligned} \tag{11.20}$$

if w is 2π-periodic and $n=0,-1,-2,\dots$. Thus (11.18) means

$$\hat{\xi}_-\left(t\right)=\sum_{n=-1}^{-\infty}C_{-n}\left(2^nt\right)2^{\frac{n}{2}}\hat{\psi}\left(2^nt\right), \tag{11.21}$$

and we have

$$m_0(Nt)\,\hat{\xi}_-(t) \sim \sum_{n=-1}^{-\infty} C_{-n}(t)\, m_0\left(N2^{-n}t\right) z^n. \tag{11.22}$$

Finally,

$$2^{\frac{1}{2}}\hat{\xi}_-(2t) \sim \sum_{n=0}^{-\infty} C_{1-n}(t)\, z^n. \tag{11.23}$$

Thus it follows from (11.16), (11.17), (11.19), (11.22), and (11.23) that

$$m_0(Nt)\, B(t) = C_1(t)$$

and

$$C_{-n}(t)\, m_0\left(N2^{-n}t\right) = C_{1-n}(t)$$

for $n = -1, -2, \dots$, i.e.,

$$C_n(t) = \prod_{k=0}^{n-1} m_0\left(N2^k t\right) \cdot B(t) \tag{11.24}$$

for $n = 1, 2, 3, \dots$. This specifies ξ_- as an A-dependent operator applied to B, and combining with (11.3) we obtain

$$\lim_{n\to\infty} 2^{-\frac{n}{2}} C_n\left(t/\left(N2^n\right)\right) = (2\pi)^{\frac{1}{2}} B(0)\,\hat{\varphi}(t), \tag{11.25}$$

with convergence in $L^2(\hat{\mathbb{R}})$.

Note also that, Fourier-transforming (11.10) using (11.11) and (11.21), we obtain the following orthogonal expansion in $L^2(\hat{\mathbb{R}})$:

$$N^{\frac{1}{2}}\hat{\varphi}(Nt) = A(t)\,\hat{\varphi}(t) + B(t)\,\hat{\psi}(t) + \sum_{n=1}^{\infty} C_n\left(2^{-n}t\right) 2^{-\frac{n}{2}}\hat{\psi}\left(2^{-n}t\right), \tag{11.26}$$

and thus

$$1 = \|A\|^2_{L^2(\mathbb{T})} + \|B\|^2_{L^2(\mathbb{T})} + \sum_{n=1}^{\infty} \|C_n\|^2_{L^2(\mathbb{T})}. \tag{11.27}$$

12. Father functions with non-orthogonal translates

It is known (see, e.g., [Dau92, Section 5.3 and Section 6.2]) that the multiresolution analysis can be extended to cases where the translates of the father function φ are not exactly orthogonal. In this section we will consider the case that (10.1) is replaced by the weaker condition that there exists a constant $c > 0$ such that

$$\left\| \sum_{n\in\mathbb{Z}} \xi_n \varphi(\,\cdot\, - n) \right\|^2_{L^2(\mathbb{R})} \leq c\,\|\xi\|^2_{\ell^2} \tag{12.1}$$

for any sequence $(\xi_n)_{n\in\mathbb{Z}}$ such that only finitely many components are nonzero. The assumptions (10.2) (scale invariance), (10.3a) (refinement), and (10.3b) (ergodicity) are kept as before. By the same reasoning leading to (10.7), condition (12.1) is equivalent to

$$\operatorname{PER}\left(|\hat{\varphi}|^2\right)(t) \leq \frac{c}{2\pi}. \tag{12.2}$$

Let us try to establish an analogue of the commutative diagram (10.28) in this more general setting. We first construct the left side of the diagram.

Lemma 12.1. *Adopt the assumptions* (12.1), (10.2), *and* (10.3). *Let μ_φ be the measure on $\mathbb{T} = \mathbb{R}\diagup 2\pi\mathbb{Z}$ with Radon-Nikodym derivative*

$$\frac{d\mu_\varphi}{dt} = \operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right). \tag{12.3}$$

Then there is a one-to-one correspondence between $f \in \mathcal{V}_0$ and $m \in L^2\left(\mathbb{T}, \mu_\varphi\right)$ given by

$$\hat{f}\left(t\right) = m\left(e^{-it}\right)\hat{\varphi}\left(t\right). \tag{12.4}$$

Moreover,

$$\left\|f\right\|_{L^2(\mathbb{R})} = \left\|m\right\|_{L^2(\mathbb{T},\mu_\varphi)}, \tag{12.5}$$

i.e., $f \to m_f$ is a unitary operator $\mathcal{V}_0 \to L^2\left(\mathbb{T}, \mu_\varphi\right)$.

Proof. Assume first that f is a finite linear combination

$$f\left(\,\cdot\,\right) = \sum_{k\in\mathbb{Z}} a_k \varphi\left(\,\cdot\, - k\right), \tag{12.6}$$

and put

$$m_f\left(\,\cdot\,\right) = m\left(\,\cdot\,\right) = \sum_k a_k e^{-ik\,\cdot}. \tag{12.7}$$

Then (12.4) is valid, and

$$\begin{aligned}\int_{\mathbb{R}} \left|\hat{f}\left(t\right)\right|^2\,dt &= \int_{\mathbb{R}} \left|m\left(e^{-it}\right)\hat{\varphi}\left(t\right)\right|^2\,dt \\ &= \int_{\mathbb{T}} \left|m\right|^2 \operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right)\,dt \\ &= \int_{\mathbb{T}} \left|m\right|^2\,d\mu_\varphi,\end{aligned}$$

so (12.5) holds. Since the set of f of the form (12.6) is dense in $\mathcal{V}_0$ by the definition of $\mathcal{V}_0$, and the set of m of the form (12.7) is dense in $L^2\left(\mathbb{T}, \mu_\varphi\right)$, the Lemma follows by closure. □

An immediate corollary is the following:

Lemma 12.2. *Adopt the assumptions* (12.1), (10.2), *and* (10.3). *Then there is a one-to-one unitary correspondence between $\psi \in \mathcal{V}_{-n} = U_N^{-n}\mathcal{V}_0$ and $m = m_{\psi,n} \in L^2\left(\mathbb{T}, \mu_\varphi\right)$ given by*

$$N^{\frac{n}{2}}\hat{\psi}\left(N^n t\right) = m\left(t\right)\hat{\varphi}\left(t\right). \tag{12.8}$$

Proof. We have $\psi \in \mathcal{V}_{-n}$ if and only if $U_N^n\psi \in \mathcal{V}_0$, and $\left\|\psi\right\|_2 = \left\|U_N^n\psi\right\|_2$. Apply Lemma 12.1 on $f = U_N^n\psi$. □

Note that (12.4) of Lemma 12.1 precisely says that the diagram

$$\begin{array}{ccc} \mathcal{V}_0 & \underset{\mathcal{F}_\varphi^{-1}}{\overset{\mathcal{F}_\varphi}{\rightleftarrows}} & L^2(\mathbb{T}, \mu_\varphi) \\ \rotatebox{-90}{$\hookrightarrow$} & & \Big\downarrow M_{\hat\varphi} \\ L^2(\mathbb{R}) & \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} & L^2(\hat{\mathbb{R}}) \end{array} \tag{12.9}$$

is commutative, where $\mathcal{F}_\varphi$ now is the map $f \to m_f$, and $M_{\hat\varphi}$ still is the map of multiplying the periodized function by $\hat\varphi$. $\mathcal{F}_\varphi$ is still unitary.

If $\psi_1, \psi_2 \in \mathcal{V}_{-1}$, and $m_i = \mathcal{F}_\varphi (U\psi_i)$, we compute

$$\begin{aligned} \langle \psi_1 \,|\, \psi_2 \rangle &= \int_{\hat{\mathbb{R}}} \bar{\hat{\psi}}_1(t)\, \hat\psi_2(t)\; dt \\ &= \frac{1}{N} \int_{\hat{\mathbb{R}}} \bar m_1(t/N)\, \bar m_2(t/N)\, \left|\hat\varphi(t/N)\right|^2\; dt \\ &= \int_{\hat{\mathbb{R}}} \bar m_1(t)\, m_2(t)\, \left|\hat\varphi(t)\right|^2\; dt \\ &= \int_{\mathbb{T}} \bar m_1(z)\, m_2(z)\; d\mu_\varphi(z)\,. \end{aligned} \tag{12.10}$$

Correspondingly,

$$\begin{aligned} \langle \psi_1 \,|\, T^k\psi_2 \rangle &= \int_{\hat{\mathbb{R}}} \bar{\hat{\psi}}_1(t)\, e^{-ikt} \hat\psi_2(t)\; dt \\ &= \frac{1}{N} \int_{\hat{\mathbb{R}}} \bar m_1(t/N)\, \bar m_2(t/N)\, e^{-ikt} \left|\hat\varphi(t/N)\right|^2\; dt \\ &= \int_{\mathbb{T}} \bar m_1(z)\, m_2(z)\, z^{kN}\; d\mu_\varphi(z)\,, \end{aligned} \tag{12.11}$$

and hence $\langle \psi_1 \,|\, T^k\psi_2 \rangle = 0$ for all $k \in \mathbb{Z}$ if and only if

$$\int_{\mathbb{T}} \bar m_1(z)\, m_2(z)\, f\left(z^N\right)\; d\mu_\varphi(z) = 0$$

for all $f \in L^\infty(\mathbb{T})$. This is equivalent to

$$\sum_{k \in \mathbb{Z}_N} \bar m_1\left(\rho^k z\right) m_2\left(\rho^k z\right) \mathrm{PER}\left(\left|\hat\varphi\right|^2\right)\left(\rho^k z\right) = 0 \tag{12.12}$$

for almost all z.

From this point one could make a similar theory as in Section 10, replacing (10.18) by (12.12) and using a selection theorem to find $m_1, \dots, m_{N-1}$, and thus $\psi_1, \dots, \psi_{N-1}$. See (12.36)–(12.37) below. However, in this case the matrix (1.11) will not be unitary, and thus the connection with representations of $\mathcal{O}_N$ is less direct. Let us rather sketch a completely different approach, where one starts with functions $m_0, m_1, \dots, m_{N-1} : \mathbb{T} \to \mathbb{C}$ such that the matrix (1.11) is assumed to be unitary at the outset. In addition we will assume that $m_0(0) = \sqrt{N}$ and that m_0 is Lipschitz continuous at 0, or merely that the infinite product

$$\hat\varphi(t) = (2\pi)^{-\frac{1}{2}} \prod_{k=1}^{\infty} \left(m_0\left(N^{-k} t\right) / N^{\frac{1}{2}}\right) \tag{12.13}$$

converges pointwise almost everywhere. By [Mal89], or [Dau92, Lemma 6.2.1], it follows from the condition $\sum_{k\in\mathbb{Z}_N} \left|m_0\left(t+2\pi k/N\right)\right|^2 = N$ that $\hat{\varphi}\in L^2\left(\mathbb{R}\right)$ and $\left\|\varphi\right\|_2\leq 1$. We will also still assume (12.1). If we now *define* $\psi_1,\dots,\psi_{N-1}$ by

$$\sqrt{N}\hat{\psi}_k\left(Nt\right) = m_k\left(t\right)\hat{\varphi}\left(t\right) \tag{12.14}$$

then

$$U^n T^k \psi_m\left(\,\cdot\,\right) = N^{-\frac{n}{2}}\psi_m\left(N^{-n}\,\cdot\,-k\right), \tag{12.15}$$

$m=1,\dots,N-1$, $n,k\in\mathbb{Z}$, no longer forms an orthonormal basis for $L^2\left(\mathbb{R}\right)$, but a *tight frame* in the sense that

$$\sum_{n,k,m}\left|\left\langle U^n T^k\psi_m \,\middle|\, f\right\rangle\right|^2 = \left\|f\right\|_2^2 \tag{12.16}$$

for all $f\in L^2\left(\mathbb{R}\right)$; see [Dau92, Proposition 6.2.3]. It is known that a tight frame is an orthonormal basis precisely when $\left\|\psi_m\right\|_2 = 1$ for $m=1,\dots,N-1$, and in general

$$f = \sum_{n,k,m}\left\langle U^n T^k\psi_m \,\middle|\, f\right\rangle U^n T^k\psi_m; \tag{12.17}$$

see [Dau92, Section 3.2].

The crucial property used in proving (12.16) as in [Dau92, Proposition 6.2.3] is the identity

$$\sum_{k\in\mathbb{Z}}\left|\left\langle U^n T^k\varphi \,\middle|\, f\right\rangle\right|^2 + \sum_{m=1}^{N-1}\sum_{k\in\mathbb{Z}}\left|\left\langle U^n T^k\psi_m \,\middle|\, f\right\rangle\right|^2 = \sum_{k\in\mathbb{Z}}\left|\left\langle U^{n-1} T^k\varphi \,\middle|\, f\right\rangle\right|^2, \tag{12.18}$$

which is verified from (12.14) and the unitarity of (1.11), and is valid for all $f\in L^2\left(\mathbb{R}\right)$. Let us check the details in the case $N=2$, where (12.18) takes the form

$$\sum_{k\in\mathbb{Z}}\left|\left\langle\varphi_{n,k}\,\middle|\,f\right\rangle\right|^2 + \sum_{k\in\mathbb{Z}}\left|\left\langle\psi_{n,k}\,\middle|\,f\right\rangle\right|^2 = \sum_{k\in\mathbb{Z}}\left|\left\langle\varphi_{n-1,k}\,\middle|\,f\right\rangle\right|^2, \tag{12.19}$$

valid for all $f\in L^2\left(\mathbb{R}\right)$. Using the argument from Theorem 9.1 adjusted as in Lemmas 12.1–12.2, we note that it is enough to check (12.19) for vectors in $U^{-j}\mathcal{V}_0$ for all $j\in\mathbb{Z}$. Note that by (12.13), the spaces $U^{-j}\mathcal{V}_0$ increase as $j\to\infty$. The vectors $f\in U^{-j}\mathcal{V}_0$ have representations as $\hat{f}\left(t\right) = 2^{-\frac{j}{2}}\left(\xi\hat{\varphi}\right)\left(t/2^j\right)$, where $\xi\in L^2\left(\mathbb{T},\mu_\varphi\right)$, according to (12.4) or (12.8). On the Fourier-transform side the terms in (12.19) then take the following form (we may assume $j\geq n$, and for simplicity, we shall do the calculation only for $n=0$, and omit the subindex n when $n=0$):

$$\begin{aligned}
\left\langle\varphi_k\,\middle|\,f\right\rangle &= \left\langle\hat{\varphi}_k\,\middle|\,\hat{f}\right\rangle\\
&= \int_{-\infty}^{\infty}\overline{\left(e_k\hat{\varphi}\right)\left(t\right)}2^{-\frac{j}{2}}\left(\xi\hat{\varphi}\right)\left(t/2^j\right)\,dt\\
&= 2^{\frac{j}{2}}\int_{-\infty}^{\infty}\overline{\left(e_k\hat{\varphi}\right)\left(2^jt\right)}\left(\xi\hat{\varphi}\right)\left(t\right)\,dt\\
&= \int_{-\infty}^{\infty}\overline{m_0^{(j)}\left(t\right)e_k\left(2^jt\right)}\left|\hat{\varphi}\left(t\right)\right|^2\xi\left(t\right)\,dt\\
&= \left\langle S_0^j e_k\,\middle|\,P\xi\right\rangle_{L^2\left(\mathbb{T}\right)},
\end{aligned}$$

where $P = 2\pi \operatorname{PER}\left(|\hat{\varphi}|^2\right)$ and $e_k(t) = e^{-ikt}$. By a similar calculation,

$$\langle \psi_k \mid f \rangle = \left\langle S_0^{j-1} S_1 e_k \,\middle|\, P\xi \right\rangle_{L^2(\mathbb{T})}$$

and

$$\langle \varphi_{-1,k} \mid f \rangle = \left\langle S_0^{j-1} e_k \,\middle|\, P\xi \right\rangle_{L^2(\mathbb{T})}.$$

Substituting back into (12.19) and using the fact that $\{e_k\}$ is an orthonormal basis for $L^2(\mathbb{T})$, we see that (12.19) just says that

$$\left\| S_0^{*\,j} P\xi \right\|_{L^2(\mathbb{T})}^2 + \left\| S_1^* S_0^{*\,j-1} P\xi \right\|_{L^2(\mathbb{T})}^2 = \left\| S_0^{*\,j-1} P\xi \right\|_{L^2(\mathbb{T})}^2,$$

which in turn takes the form

$$P S_0^{j-1} \left(S_0 S_0^* + S_1 S_1^* \right) S_0^{*\,j-1} P = P S_0^{j-1} S_0^{*\,j-1} P,$$

and this follows immediately from $S_0 S_0^* + S_1 S_1^* = I_{L^2(\mathbb{T})}$. This proves (12.18), and thus $\left\{ U^n T^k \psi_m \right\}$ forms a tight frame. (Compare the present argument to the one of the proof of Corollary 10.4, and to (1.33) and (1.35) in Section 1.)

Remark 12.3. An alternative way of defining a commutative diagram like (12.9) is the following. Define a map $\mathcal{F}_\varphi : \mathcal{V}_0 \to L^2(\mathbb{T}, \mu_\varphi)$, *different* from the $\mathcal{F}_\varphi$ defined after (12.9), by

$$(\mathcal{F}_\varphi f)\left(e^{-it}\right) = \sum_{k \in \mathbb{Z}} \langle \varphi(\,\cdot\, - k) \mid f \rangle e^{-ikt}. \tag{12.20}$$

Then

$$\begin{aligned} \|\mathcal{F}_\varphi f\|_2^2 &= \sum_k \left| \langle \varphi(\,\cdot\, - k) \mid f \rangle \right|^2 \\ &= \sum_k \left| \left\langle e^{-ik\,\hat{\cdot}}\, \hat{\varphi}(\,\hat{\cdot}\,) \,\middle|\, m_f\left(e^{-i\,\hat{\cdot}}\right) \hat{\varphi}(\,\hat{\cdot}\,) \right\rangle \right|^2 \\ &= \sum_k \left| \int_{\mathbb{T}} e^{ikt} m_f\left(e^{-ikt}\right) \operatorname{PER}\left(|\hat{\varphi}|^2\right)(t)\, dt \right|^2 \\ &= (2\pi)^2 \int_{\mathbb{T}} \left| m_f \operatorname{PER}\left(|\hat{\varphi}|^2\right) \right|^2, \end{aligned} \tag{12.21}$$

where the Haar measure $dt/2\pi$ on $\mathbb{T}$ is implicit. On the other hand,

$$\begin{aligned} \|f\|_2^2 &= \int_{\mathbb{R}} |m_f \hat{\varphi}|^2 (t)\, dt \\ &= 2\pi \int_{\mathbb{T}} |m_f|^2 \operatorname{PER}\left(|\hat{\varphi}|^2\right). \end{aligned} \tag{12.22}$$

Thus, using (12.1) in the form (12.2) we have

$$\|\mathcal{F}_\varphi f\|_2^2 \le c\, \|f\|_2^2, \tag{12.23}$$

so $\mathcal{F}_\varphi : \mathcal{V}_0 \to L^2(\mathbb{T}, \mu_\varphi)$ is bounded. Next, define a map $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\hat{\mathbb{R}})$ as a modified Fourier transform:

$$\mathcal{F}(f) = 2\pi \operatorname{PER}\left(|\hat{\varphi}|^2\right) \hat{f} = P\hat{f}, \tag{12.24}$$

where $P = 2\pi \operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right)$. The reason for this definition is the following computation, valid for $f \in \mathcal{V}_0$, i.e., $\hat{f} = m_f \hat{\varphi}$:

$$(M_{\hat{\varphi}} \mathcal{F}_\varphi f)(\,\hat{\cdot}\,) = \sum_{k\in\mathbb{Z}} \langle \varphi(\,\cdot\, - k) \mid f \rangle \, \hat{\varphi}(\,\hat{\cdot}\,)\, e^{-ik\,\hat{\cdot}}$$
$$= \sum_{k\in\mathbb{Z}} \left\langle e^{-ik\,\hat{\cdot}}\, \hat{\varphi} \,\middle|\, m_f \hat{\varphi} \right\rangle \hat{\varphi}(\,\hat{\cdot}\,)\, e^{-ik\,\hat{\cdot}}$$
$$= \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}} e^{ikt} m_f(t) \left|\hat{\varphi}\right|^2(t)\, dt \right) \hat{\varphi}(\,\hat{\cdot}\,)\, e^{-ik\,\hat{\cdot}}$$
$$= \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{T}} e^{ikt} m_f(t) \operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right)(t)\, dt \right) e^{-ik\,\hat{\cdot}}\, \hat{\varphi}(\,\hat{\cdot}\,)$$
$$= m_f(\,\hat{\cdot}\,)\, P(\,\hat{\cdot}\,)\, \hat{\varphi}(\,\hat{\cdot}\,)$$
$$= P(\,\hat{\cdot}\,)\, \hat{f}(\,\hat{\cdot}\,)$$
$$= (\mathcal{F} f)(\,\hat{\cdot}\,). \tag{12.25}$$

Thus, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{V}_0 & \xrightarrow{\mathcal{F}_\varphi} & L^2(\mathbb{T}, \mu_\varphi) \\ \big\downarrow & & \big\downarrow M_{\hat{\varphi}} \\ L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\hat{\mathbb{R}}) \end{array} \tag{12.26}$$

This new diagram should not be confused with (12.9), as the maps are defined differently. The new maps $\mathcal{F}_\varphi$ and $\mathcal{F}$ are no longer isometries, but merely continuous, and they are invertible if and only if there is a lower estimate

$$b \left\|\xi\right\|_{\ell^2}^2 \leq \left\| \sum_{n\in\mathbb{Z}} \xi_n \varphi(\,\cdot\, - n) \right\|_{L^2(\mathbb{R})}^2 \tag{12.27}$$

or, equivalently,

$$\frac{b}{2\pi} \leq \operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right)(t). \tag{12.28}$$

After this digression, we let $\mathcal{F}$ and $\mathcal{F}_\varphi$ have the same meaning as in (12.9) in the rest of the discussion. For the same reason as above, the map

$$\operatorname{id}_\varphi : L^2(\mathbb{T}) \to L^2(\mathbb{T}, \mu_\varphi) \tag{12.29}$$

given by $f \to f$ is bounded and of norm at most c if and only if (12.2) is valid, and then $\operatorname{id}_\varphi^{-1}$ exists as a bounded operator if and only if (12.28) holds.

We now define

$$J : L^2(\hat{\mathbb{R}}) \longrightarrow \mathbb{C}^{N-1} \otimes \mathcal{K} \otimes L^2(\mathbb{T}) \tag{12.30}$$

differently from the J in (10.27), by

$$J\left(\check{f}\right)\left(e^{-it}, z\right) = \sum_{m=1}^{N-1} \sum_{n,k\in\mathbb{Z}} \left\langle U^n T^k \psi_m \,\middle|\, f \right\rangle e_m \otimes e^{-ikt} \otimes z^n, \tag{12.31}$$

where $\check{f} = \mathcal{F}^{-1} f$ is the inverse Fourier transform of f. The map J, so defined, is an isometry because of (12.16), but it is not necessarily surjective, and (12.31) coincides with (10.27) if and only if $\left\{U^n T^k \psi_m\right\}$ is an orthonormal basis. The intertwining property (9.29) also carries over to the present more general setting, and it takes the form

$$J\mathcal{F}U = M_z J\mathcal{F}. \tag{12.32}$$

Let us now do the simple computation of this, omitted in (9.29), in the case $N = 2$. Putting $\hat{J} = J\mathcal{F}$, we have

$$\hat{J} f = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left\langle \psi_{n,k} \,\middle|\, f \right\rangle_{L^2(\mathbb{R})} e_k z^n. \tag{12.33}$$

We must show

$$M_z \hat{J} = \hat{J} U, \tag{12.34}$$

where M_z again is the operator on $\mathcal{K} \otimes L^2(\mathbb{T})$ given by $\left(M_z \xi\right)(z) = z\xi(z)$, $z \in \mathbb{T}$.

We now check (12.34): let $f \in L^2(\mathbb{R})$. Then by (12.33), using temporarily the notation $e_k(t) = e^{-ikt}$,

$$\begin{aligned}
\left(\hat{J} U f\right)(\,\cdot\,, z) &= \sum_n \sum_k \left\langle \psi_{n,k} \,\middle|\, U f \right\rangle e_k z^n \\
&= \sum_n \sum_k 2^{\frac{1+n}{2}} \int_{-\infty}^{\infty} \overline{\left(e_k \hat{\psi}\right)(2^n t)} \hat{f}(2t)\, dt\, e_k z^n \\
&= \sum_n \sum_k 2^{\frac{n-1}{2}} \int_{-\infty}^{\infty} \overline{\left(e_k \hat{\psi}\right)(2^{n-1} t)} \hat{f}(t)\, dt\, e_k z^n \\
&= \sum_n \sum_k \left\langle \psi_{n-1,k} \,\middle|\, f \right\rangle e_k z^n \\
&= \sum_n \sum_k \left\langle \psi_{n,k} \,\middle|\, f \right\rangle e_k z^{n+1} \\
&= \left(M_z \hat{J} f\right)(\,\cdot\,, z),
\end{aligned}$$

and this completes the proof of (12.34).

If the operator $V : \mathbb{C}^{N-1} \otimes \mathcal{K} \otimes H_+^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined exactly as before in (6.14) and (10.25), it is still unitary. The diagram corresponding to the right-hand side of (10.28) is

$$\begin{array}{ccccc}
L^2(\mathbb{T}, \mu_\varphi) & \xleftarrow{\mathrm{id}_\varphi} & \mathcal{K} = L^2(\mathbb{T}) & \xleftarrow{V} & \mathbb{C}^{N-1} \otimes \mathcal{K} \otimes H_+^2(\mathbb{T}) \\
\Big\downarrow{\scriptstyle M_{\hat{\varphi}}} & & & & \big\downarrow \\
L^2(\hat{\mathbb{R}}) & & \xrightarrow{\qquad\qquad J \qquad\qquad} & & \mathbb{C}^{N-1} \otimes \mathcal{K} \otimes L^2(\mathbb{T})
\end{array} \tag{12.35}$$

This diagram is necessarily not commutative, however, unless $\mathrm{PER}\left(|\hat{\varphi}|^2\right) = \frac{1}{2\pi}$, i.e., (10.1) is fulfilled. The reason is that the maps V, $M_{\hat{\varphi}}$, J, and the inclusion map are all isometries, while id_φ is not unless $\mathrm{PER}\left(|\hat{\varphi}|^2\right) = \frac{1}{2\pi}$. In order to make the diagram commutative, V would have to be redefined. One way would be to

choose $m_1, \dots, m_{N-1} \in L^\infty(\mathbb{T})$ such that the relations

$$\sum_{k\in\mathbb{Z}_N} \bar{m}_i\left(\rho^k z\right) m_j\left(\rho^k z\right) \operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right)\left(\rho^k z\right) 2\pi = N\delta_{ij} \tag{12.36}$$

are valid for almost all z (see (12.10)–(12.12)), and then define S_k on $L^2(\mathbb{T};\mu_\varphi)$ by

$$\left(S_k\xi\right)(z) = m_k(z)\,\xi\left(z^N\right). \tag{12.37}$$

One verifies that this defines a representation of $\mathcal{O}_N$, and hence the corresponding V is a unitary. These remaining details for making a commutative variant of the diagram will be published in a forthcoming paper.

Let us now consider further the orthogonality properties of the $\mathbb{Z}$-translates $\{\varphi(\,\cdot\,-k)\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{R})$.

Let $\varphi \in L^2(\mathbb{R})$, and assume that φ can be expanded like

$$U\varphi = \sum_k a_k \varphi(\,\cdot\,-k), \tag{12.38}$$

where $\sum_k |a_k|^2 < \infty$, and where

$$(U\varphi)(x) = N^{-\frac{1}{2}}\varphi(x/N). \tag{12.39}$$

Thus

$$\sqrt{N}\hat{\varphi}(Nt) = m_0(t)\,\hat{\varphi}(t), \tag{12.40}$$

where

$$m_0(t) = \sum_k a_k e^{-ikt}, \tag{12.41}$$

so $m_0 \in L^2(\mathbb{T})$.

From now and through the rest of this section, we will make the overall assumption that m_0 is uniformly Lipschitz continuous, i.e., there exists a $K > 0$ such that $|m_0(t) - m_0(s)| \le K|t-s|$ for all $t, s \in \mathbb{R}$. This condition is for example implied by the stronger condition $\sum_k |ka_k| < \infty$ which is much used in [Dau92]. We may then define an operator $R : C(\mathbb{T}) \to C(\mathbb{T})$ by

$$(R\xi)(z) = \frac{1}{N}\sum_{\substack{w\\ w^N=z}} |m_0(w)|^2\,\xi(w). \tag{12.42}$$

Proposition 12.4. *If m_0 is uniformly Lipschitz continuous, $\hat{\varphi}$ is continuous at 0, and φ and m_0 are normalized by $\hat{\varphi}(0) = (2\pi)^{-\frac{1}{2}}$ and $m_0(0) = N^{\frac{1}{2}}$ so that $\hat{\varphi}(t) = (2\pi)^{-\frac{1}{2}}\prod_{k=1}^\infty\left(N^{-\frac{1}{2}}m_0\left(tN^{-k}\right)\right)$, then the following conditions are equivalent.*

(12.43) *$\{\varphi(\,\cdot\,-k)\}$ is an orthonormal set.*

(12.44) $\operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right) = (2\pi)^{-1}\,\mathbb{1}$.

(12.45) *Up to a scalar, $\mathbb{1}$ is the unique eigenvector of R of eigenvalue 1.*

Proof. We already proved the implications (12.43)⇔(12.44)⇐(12.45) in the remarks around (10.6)–(10.10). In particular, (12.45) and (10.8) imply that $\operatorname{PER}\left(\left|\hat{\varphi}\right|^2\right)$ is

a scalar multiple of 1, and hence the $\{\varphi(\cdot - k)\}$ are orthogonal, but then, as

$$\frac{1}{N}\sum_{\substack{w \\ w^N=z}} |m_0(w)|^2 = 1$$

as a consequence of (12.45) and $m_0(0) = N^{\frac{1}{2}}$, we have $m_0\left(2\pi\frac{k}{N}\right) = 0$ for $k = 1, \dots, N-1$, and it follows from the product expansion of $\hat{\varphi}$ that $\hat{\varphi}(2\pi k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. Thus $\mathrm{PER}\left(|\hat{\varphi}|^2\right)(0) = (2\pi)^{-1}$, and as $\mathrm{PER}\left(|\hat{\varphi}|^2\right)$ is a scalar multiple of $\mathbb{1}$, (12.44) follows. Implicit in this reasoning is that $\mathrm{PER}\left(|\hat{\varphi}|^2\right)$ is continuous on $\mathbb{T}$, but this follows from the uniform Lipschitz condition on m_0 and the product expansion. It remains to prove (12.43)⇒(12.45). It follows from (12.44) (⇔(12.43)) and (10.8) that $\mathbb{1}$ is indeed an eigenvector of R of eigenvalue 1, and it remains to show that it is the only one. (See Remark 12.7.)

Lemma 12.5. *Let $N \in \mathbb{N}$, $N \geq 2$, and $m_0 \in L^\infty(\mathbb{T})$ be given, satisfying*

$$\sum_{\substack{w \\ w^N=z}} |m_0(w)|^2 = N, \tag{12.46}$$

for almost all $z \in \mathbb{T}$. Let S_0 be the corresponding isometry of $L^2(\mathbb{T})$,

$$(S_0 f)(z) = m_0(z) f\left(z^N\right), \quad f \in L^2(\mathbb{T}).$$

Let m_0 and φ further satisfy the general conditions in Proposition 12.4. *Then the orthogonality condition* (12.43) *for φ in $L^2(\mathbb{R})$ is equivalent to*

$$\lim_{n\to\infty} \langle \mathbb{1} \,|\, S_0^{*\,n} M_f S_0^n \mathbb{1}\rangle_{L^2(\mathbb{T})} = f(0) \tag{12.47}$$

for all $f \in C(\mathbb{T}) = C(\mathbb{R}/2\pi\mathbb{Z})$. Thus the two conditions (12.46) *and* (12.47) *together are equivalent to the other conditions in Proposition* 12.4.

In general, if we do not assume the orthogonality (12.43) but merely its consequence (12.46), the left-hand limit in (12.47) exists and defines a probability measure D on $\mathbb{T}$. The fact that a Borel measure D is defined as

$$\begin{aligned} D(f) &= \lim_{n\to\infty} \langle \mathbb{1} \,|\, S_0^{*\,n} M_f S_0^n \mathbb{1}\rangle \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} f\, d\nu_n \end{aligned} \tag{12.48}$$

is justified in the discussion of Remark 12.6 below. It is a compactness argument, referring to the Hausdorff metric on the Borel probability measures on $\mathbb{T}$, and it requires the Lipschitz assumption on the function m_0; see also [Hut81] for definitions. If μ and ν are Borel measures on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, then the Hausdorff metric d_H is

$$d_H(\mu,\nu) := \sup_f \left\{ \left| \int_{\mathbb{T}} f\, d\mu - \int_{\mathbb{T}} f\, d\nu \right| \;\middle|\; f \in C^1(\mathbb{T}), \sup_t |f'(t)| \leq 1 \right\}$$

with C^1-functions on $\mathbb{T}$ identified with differentiable 2π-periodic functions on $\mathbb{R}$. The approximation $\nu_n \to D$ in (12.48) refers to $\lim_{n\to\infty} d_H(\nu_n, D) = 0$.

Proof of Lemma 12.5. We use the result of Meyer and Paiva [MePa93] mentioned in (1.45) to the effect that (12.43) is equivalent in turn to

$$\int_{\delta\leq|t|\leq\pi} P_n(t)\,dt \underset{n\to\infty}{\longrightarrow} 0, \tag{12.49}$$

for all positive δ, where (by identification) $m_0(t) \sim m_0\left(e^{-it}\right)$, and

$$P_n(t) := \left|m_0(t)\,m_0(Nt)\cdots m_0\left(N^{n-1}t\right)\right|^2.$$

The lemma follows from Meyer–Paiva using

$$\begin{aligned}\left(S_0^n\mathbb{1}\right)(z) &= m_0(z)\,m_0\left(z^N\right)\cdots m_0\left(z^{N^{n-1}}\right)\\ &=: m_0^{(n)}(z),\\ P_n(t) &= \left|m_0^{(n)}\left(e^{-it}\right)\right|^2,\end{aligned}$$

thus,

$$\begin{aligned}\left\langle S_0^n\mathbb{1}\,\middle|\,fS_0^n\mathbb{1}\right\rangle &= \left\langle\mathbb{1}\,\middle|\,S_0^{*\,n}M_fS_0^n\mathbb{1}\right\rangle\\ &= \int_{\mathbb{T}}\left(R^nf\right)(z)\,\frac{|dz|}{2\pi}\\ &= \int_{\mathbb{T}}\left|m_0^{(n)}(z)\right|^2 f(z)\,\frac{|dz|}{2\pi}\\ &= \int_{\mathbb{T}}P_n(z)\,f(z)\,\frac{|dz|}{2\pi},\end{aligned}$$

and an elementary characterization of the Dirac mass at $z=1$. ☐

To prove (12.43)⇒(12.45), it thus suffices to show that (12.46) and (12.47) imply (12.45). To this end, one easily deduces from (3.20) that

$$S_0^{*\,n}M_fS_0^n = M_{R^nf}, \tag{12.50}$$

where still

$$\left(Rf\right)(z) = N^{-1}\sum_{\substack{w\\ w^N=z}}\left|m_0(w)\right|^2 f(w).$$

We conclude from (12.47) that

$$f(0) = \lim_{n\to\infty}\int_{\mathbb{T}}\left(R^nf\right)(z)\,\frac{|dz|}{2\pi} \tag{12.51}$$

for all $f\in C(\mathbb{T})$. Note the formula

$$\left(R^nf\right)(z) = N^{-n}\sum_{\substack{w\\ w^{N^n}=z}}P_n(w)\,f(w). \tag{12.52}$$

It follows from (12.51) that

$$f(0) = \int_{\mathbb{T}}f(z)\,\frac{|dz|}{2\pi} \tag{12.53}$$

if f satisfies

$$Rf = f. \tag{12.54}$$

From (12.54), we obtain by the Schwarz inequality (see [BrRo96, Notes and Remarks to Section 5.3.1]) applied to R:

$$|f|^2 = |Rf|^2 \leq R\left(|f|^2\right).$$

By induction, then,

$$|f|^2 \leq R^n\left(|f|^2\right) \leq R^{n+1}\left(|f|^2\right),$$

and therefore, from (12.51),

$$|f(0)|^2 = \lim_{n\to\infty} \int_{\mathbb{T}} R^n\left(|f|^2\right) \geq \int_{\mathbb{T}} |f|^2,$$

where the Haar measure $\frac{|dz|}{2\pi}$ is implicitly understood. Using Cauchy-Schwarz and (12.53), we then conclude that

$$|f(0)|^2 = \left|\int_{\mathbb{T}} f\right|^2 \leq \int_{\mathbb{T}} |f|^2 \leq |f(0)|^2. \quad \text{(Recall } z = e^{-it}.\text{)}$$

We conclude that the Cauchy–Schwarz inequality is an equality when applied to the two functions f and $\mathbb{1}$. Hence f is a constant multiple of $\mathbb{1}$, and we have proved that the eigenspace of R corresponding to the eigenvalue 1 is one-dimensional. This ends the proof of Proposition 12.4. □

Remark 12.6. Using (12.46), we conclude that the measure D defined in general by (12.48) satisfies the two invariance properties below (12.55)–(12.56), even when (12.43) is not assumed:

$$D(R(f)) = D(f) \tag{12.55}$$

for all $f \in C(\mathbb{T})$, and

$$D(\sigma(f)) = D(f) \tag{12.56}$$

where $\sigma(f)(z) = f\left(z^N\right)$. Since σ is mixing (see [Kea72]), the measure D (in the wavelet examples) must be singular, but with support on $\mathbb{T}$ invariant under σ.

The measure D defined by (12.48) exists by the Ruelle-Perron-Frobenius theorem in the form of [PaPo90, p. 21], at least if $m_0(z) \neq 0$ for all z, except for a finite number of zeroes of m_0; see [Kea72] or [Hut81]. The required regularity condition on $m_0 = \sum_{k\in\mathbb{Z}} a_k z^k$ is $\sum_{k\in\mathbb{Z}} |k a_k| < \infty$, which also guarantees convergence of the infinite product formula for $\hat{\varphi}$. Since $|m_0(z)|^2 P_n\left(z^N\right) = P_{n+1}(z)$, the invariance (12.56) of the measure D follows: specifically,

$$\begin{aligned}\int_{\mathbb{T}} P_{n+1}(z) f\left(z^N\right) \frac{|dz|}{2\pi} &= \int_{\mathbb{T}} |m_0(z)|^2 P_n\left(z^N\right) f\left(z^N\right) \frac{|dz|}{2\pi} \\ &= \frac{1}{N} \int_{\mathbb{T}} \sum_{\substack{w \\ w^N = z}} |m_0(w)|^2 P_n(z) f(z) \frac{|dz|}{2\pi} \\ &= \int_{\mathbb{T}} P_n(z) f(z) \frac{|dz|}{2\pi},\end{aligned}$$

so (12.56) follows upon taking the $n \to \infty$ limit.

Using the estimate

$$N^n \left|\hat{\varphi}(N^n t)\right|^2 \leq \frac{1}{2\pi} P_n\left(e^{-it}\right), \tag{12.57}$$

which follows from (12.13), (12.46), and the definition of P_n after (12.49), we will argue that

$$\|\varphi\|^2_{L^2(\mathbb{R})} \leq D\left(\{0\}\right). \tag{12.58}$$

Integrating (12.57) over $\langle -\eta, \eta\rangle$, where $\eta > 0$, we obtain

$$\int_{-\eta N^n}^{\eta N^n} \left|\hat{\varphi}\left(t\right)\right|^2 \, dt \leq \int_{|t|<\eta} P_n\left(t\right) \, dt. \tag{12.59}$$

The limit on the left-hand side (as $n \to \infty$) is

$$\int_{\mathbb{R}} \left|\hat{\varphi}\right|^2 \, dt = \|\varphi\|^2_{L^2(\mathbb{R})},$$

and on the right it is $D\left(\langle -\eta, \eta\rangle\right)$ by formula (12.48) applied to $f = \chi_{\langle -\eta,\eta\rangle}$. Letting $\eta \to 0$, (12.58) follows. Using (12.55) and (12.56), we note that when $\operatorname{supp}\left(D\right)$ is finite, then there are cycles corresponding to roots $a \in \mathbb{T}$ of $a^{N^k} = a$, $k = 1, 2, \dots$ (k chosen minimal), such that D is a convex combination of associated measures D_a defined as

$$D_a := \frac{1}{k} \sum_{j=0}^{k-1} \delta_{a^{N^j}},$$

where $\delta_{a^{N^j}}$ denotes the Dirac mass at the point $z = a^{N^j}$ on $\mathbb{T}$. The case $D_1 = \delta_1$ may occur in the convex combination because $\left|m_0\left(1\right)\right| = \sqrt{N}$ in our case, referring to the z-parameter on $\mathbb{T}$. By (12.55), a cycle D_a may in general occur in the convex combination for D iff $\left|m_0\left(a^{N^j}\right)\right| = \sqrt{N}$ for $j = 0, 1, \dots, N^{k-1}$.

Recall that we have encountered the functions $P_n\left(t\right)$ before in a situation where the normalization $m_0\left(0\right) = N^{\frac{1}{2}}$ is not fulfilled, in the proof of Lemma 3.2.

The finite-orbit picture for the $z \mapsto z^N$ action on $\operatorname{supp}\left(D\right)$ ($\subset \mathbb{T}$), and its connection to the Cohen cycles (see [Coh90] and [Dau92, Theorem 6.3.3, p. 188]), will be taken up in a subsequent paper. This decomposition is also closely connected (in a special case) to one which arises in an earlier paper of ours [BrJo96b, Proposition 8.2].

In a forthcoming paper, we plan to study the other possibilities for $\operatorname{supp}\left(D\right)$: possibly infinite, possibly allowing infinite orbits, or an infinite number of finite orbits, under the restricted action of $z \mapsto z^N$ on $\operatorname{supp}\left(D\right)$.

Remark 12.7. Let us check how much mileage we can get towards the proof of (12.43)⇒(12.45) in Proposition 12.4 without using Lemma 12.5. For this, let $\xi \in$

$L^{\infty}(\mathbb{T})$ and $\eta \in L^2(\mathbb{T})$, and compute

$$\begin{aligned}
\int_{\mathbb{T}} (R\xi)(z)\,\eta(z)\,\frac{|dz|}{2\pi} &= \int_{\mathbb{T}} \frac{1}{N} \sum_{\substack{w \\ w^N = z}} |m_0(w)|^2\,\xi(w)\,\eta\left(w^N\right)\frac{|dz|}{2\pi} \\
&= \int_{\mathbb{T}} |m_0(z)|^2\,\xi(z)\,\eta\left(z^N\right)\frac{|dz|}{2\pi} && \text{(by (3.1))} \\
&= \int_{\mathbb{R}} |m_0(t)|^2\,|\hat{\varphi}(t)|^2\,\xi(t)\,\eta(Nt)\;dt && \text{(by (12.44))} \\
&= N\int_{\mathbb{R}} \xi(t)\,|\hat{\varphi}(Nt)|^2\,\eta(Nt)\;dt && \text{(by (12.40))} \\
&= \int_{\mathbb{R}} \xi(t/N)\,|\hat{\varphi}(t)|^2\,\eta(t)\;dt.
\end{aligned}$$

If ξ is an eigenvector for R with eigenvalue 1, this computation gives

$$\int_{\mathbb{R}} \left(\xi(t) - \xi(t/N)\right)|\hat{\varphi}(t)|^2\,\eta(t)\;dt = 0$$

for all $\eta \in L^2(\mathbb{T})$. Conversely, one checks that this N-scale condition on ξ implies that ξ is an eigenvector for R of eigenvalue 1.

13. Concluding remarks

Operators of the form (1.8) or (1.16) occur in a variety of contexts: for example, in Ruelle's work on dynamical systems [Rue94], [BaRu96]; as operators in spaces of analytic functions [CoMa95], [Ho96], [Lam86], [HoJa96], [LaSt91] under the names "weighted translation operators", "composition operators", or "slash Toeplitz operators"; and in ergodic theory [Kea72], [Wal96] (in the positive case). Our present approach is different from those mentioned in that we ask the questions in a geometric Hilbert-space setting in $L^2(\mathbb{T})$, and in that we make the connections between wavelets and the theory of representations of the C^*-algebras $\mathcal{O}_N$. Our analysis of the mother functions $\psi_1, \dots, \psi_{N-1}$ is motivated by results in [GrMa92], [GrHa94], and [Mey93, Mey92], while our study of the correspondence between the $\mathcal{O}_N$-representations and $\{\varphi, \psi_i\}$, in (1.39)–(1.40), is motivated by our desire to put the results of [CoRy95] and [CoDa96] in a more general geometric and operator-theoretic framework. Our viewpoint here, and in particular in Section 11, is that of Lax and Phillips [LaPh89] in their approach to scattering on obstacles for the classical wave equation.

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